Twist (6D Velocity Parameterization)

Setup

- Let us first parameterize the motion of a body frame by time:
  - An observer associated to $\mathscr{F}_o$ records the motion as $T^o_{s'\to b(t)}$, where the body frame is at $\mathscr{F}_{b(t)}$.

Twist

$$\begin{aligned} T^o_{s'\to b(t+\Delta t)} - T^o_{s'\to b(t)} &= T^o_{b(t)\to b(t+\Delta t)} T^o_{s'\to b(t)} - T^o_{s'\to b(t)} \\ &= e^{[\chi^o_{b(t)\to b(t+\Delta t)}]} T^o_{s'\to b(t)} - T^o_{s'\to b(t)} \\ &\approx [\chi^o_{b(t)\to b(t+\Delta t)}] T^o_{s'\to b(t)} \end{aligned}$$

- Divided by Δt and take the limit, we have

$$\begin{aligned} \dot{T}^o_{s'\to b(t)} &= \lim_{\Delta t\to 0} \left[\frac{\chi^o_{b(t)\to b(t+\Delta t)}}{\Delta t} \right] T^o_{s'\to b(t)} \\ &= [\xi^o_{b(t)}] T^o_{s'\to b(t)} \end{aligned}$$

- $\xi^o_{b(t)} := \lim_{\Delta t\to 0} \dfrac{\chi^o_{b(t)\to b(t+\Delta t)}}{\Delta t}$ is called "**twist**", the 6D instant velocity

Twist

- Twist: $\xi^{o}_{b(t)} := \lim_{\Delta t \to 0} \frac{\chi^{o}_{b(t) \to b(t+\Delta t)}}{\Delta t}$
- $[\xi^{o}_{b(t)}] = \dot{T}^{o}_{s' \to b(t)} (T^{o}_{s' \to b(t)})^{-1}$
- Note: $\xi^{o}_{b(t)} \neq \dot{\chi}^{o}_{s' \to b(t)}$ for general $\chi^{o}_{s \to b(t)}(t)$ (verify by yourself)

Linear Velocity from Twist

- The linear velocity of p^o caused by $T^o_{s' \to b(t)}$ at time t is

$$\mathbf{v}^o_p(t) = \lim_{\Delta t \to 0} \frac{T^o_{b(t)\to b(t+\Delta t)} p^o - p^o}{\Delta t} = \lim_{\Delta t \to 0} \frac{\exp([\chi^o_{b(t)\to b(t+\Delta t)}]) - I}{\Delta t} p^o$$

$$= \lim_{\Delta t \to 0} \frac{[\chi^o_{b(t)\to b(t+\Delta t)}]}{\Delta t} p^o = [\xi^o_{b(t)}] p^o$$

- Therefore, $\boxed{\mathbf{v}^o_p(t) = [\xi^o_{b(t)}] p^o}$

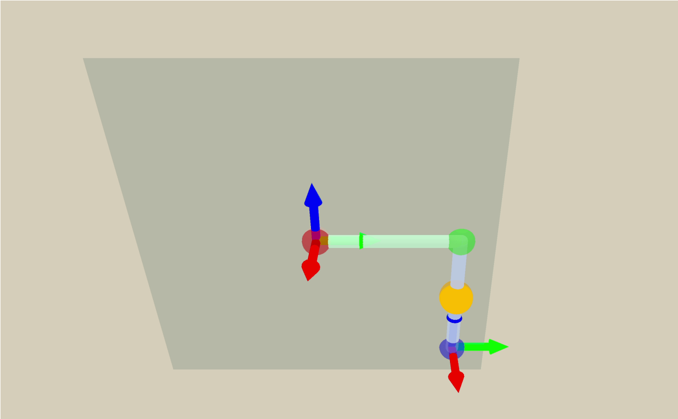
(Recall that, if a motion is a pure rotation, then $\mathbf{v}^o_p(t) = \omega^o_{b(t)} \times p^o$)

Example of Twist Computation

Example of Twist Computation

- Consider the example, but now an orange point is fixed to the end-effector frame (blue sphere)
- What is the **velocity of orange point at** $t = 0$? Given the pose of end effector frame as below:

$$T^{s}_{s \to b(t)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\alpha t) & -\sin(\alpha t) & 1 + \sin(\alpha t) \\ 0 & \sin(\alpha t) & \cos(\alpha t) & -\cos(\alpha t) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Example of Twist Computation

- The velocity of yellow point caused by the end-effector motion can be computed via twist
- Recall: $\mathbf{v}_p^s = [\xi_{b(t)}^s]p^s$

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- By $T_{s\to b(t)}^s = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\alpha t) & -\sin(\alpha t) & 1+\sin(\alpha t) \\ 0 & \sin(\alpha t) & \cos(\alpha t) & -\cos(\alpha t) \\ 0 & 0 & 0 & 1 \end{bmatrix}, \dot{T}_{s\to b(t)}^s = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -\sin(\alpha t) & -\cos(\alpha t) & \cos(\alpha t) \\ 0 & \cos(\alpha t) & -\sin(\alpha t) & \sin(\alpha t) \\ 0 & 0 & 0 & 0 \end{bmatrix},$
- we have $[\xi_{s\to b(t)}^s] = \dot{T}_{s\to b(t)}^s (T_{s\to b(t)}^s)^{-1} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\alpha & 0 \\ 0 & \alpha & 0 & -\alpha \\ 0 & 0 & 0 & 0 \end{bmatrix}$

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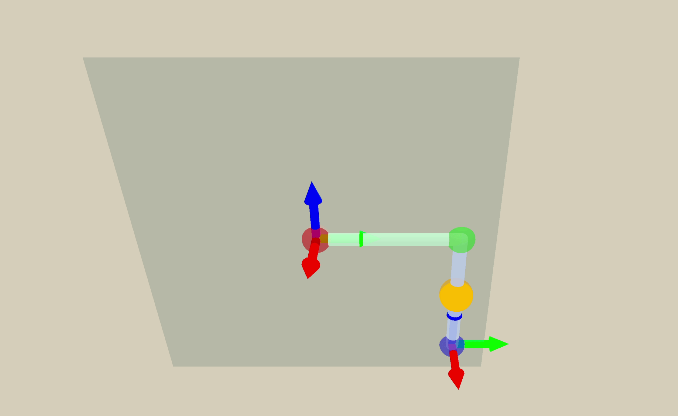
At $t = 0, p^s = \begin{bmatrix} 0 \\ 1 \\ -\frac{1}{2} \\ 1 \end{bmatrix}$



Example of Twist Computation

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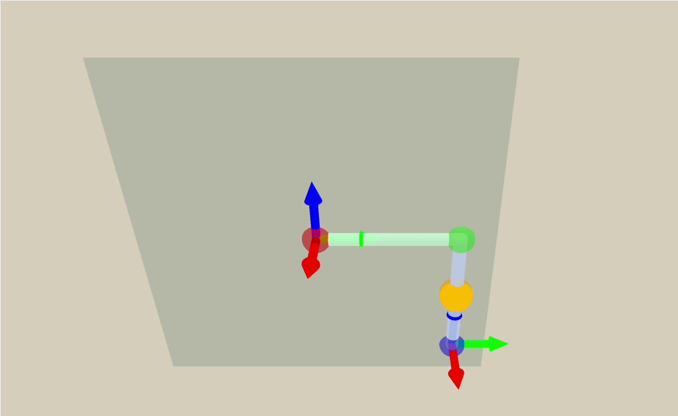
$$[\xi_{b(t)}^s] = \dot{T}_{s\to b(t)}^s(T_{s\to b(t)}^s)^{-1} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\alpha & 0 \\ 0 & \alpha & 0 & -\alpha \\ 0 & 0 & 0 & 0 \end{bmatrix}, p^s = \begin{bmatrix} 0 \\ 1 \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

$$\mathbf{v}_p^s = [\xi_{b(t)}^s]p^s = \begin{bmatrix} 0 \\ \frac{\alpha}{2} \\ 0 \\ 0 \end{bmatrix}$$

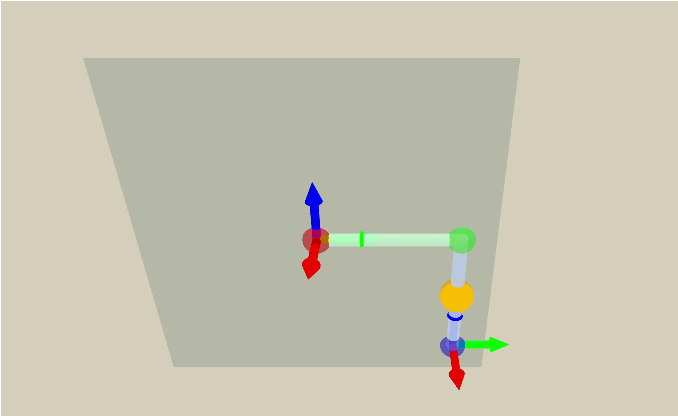


Example of Twist Computation

- We can verify this result by taking the derivative of $\frac{d}{dt}p^s(t)$

$$p^s(t) = \begin{bmatrix} 0 \\ 1 + \frac{1}{2}\sin(\alpha t) \\ -\frac{1}{2}\cos(\alpha t) \\ 1 \end{bmatrix}, \quad \frac{d}{dt}p^s(t) = \begin{bmatrix} 0 \\ \frac{\alpha}{2}\cos(\alpha t) \\ \frac{\alpha}{2}\sin(\alpha t) \\ 0 \end{bmatrix}$$

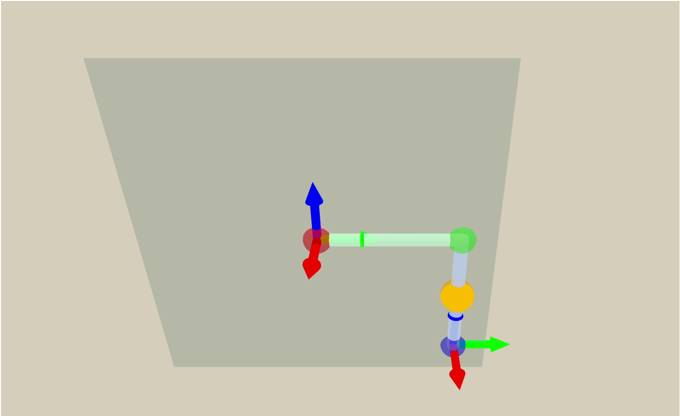
$$\mathbf{v}_p^s = [\xi_{b(t)}^s]p^s = \begin{bmatrix} 0 \\ \frac{\alpha}{2} \\ 0 \\ 0 \end{bmatrix} = \frac{d}{dt}p^s(t)\Big|_{t=0}$$



Example of Twist Computation

- What is the body twist of the end effector?
- In the body frame of the end effector (blue sphere), the origin of the frame, which is the blue sphere, has a constant linear velocity, which is always $[0, \alpha, 0]$. The angular velocity is always $[\alpha, 0, 0]$.

So, $\xi_{b(t)}^{b(t)} = [0, \alpha, 0, \alpha, 0, 0]^T$



Change of Coordinates for Twists

Review

- Recall that, the recordings by different observers are related by the similarity transformation:

$$T^{s_1}_{1\to 2} = T_{s_1\to s_2} T^{s_2}_{1\to 2} (T_{s_1\to s_2})^{-1}$$

Tricks in Recording Velocities

- If transformations could be recorded differently by observers, velocity should also be recorded differently

Relating 6D Velocities from Different Observers

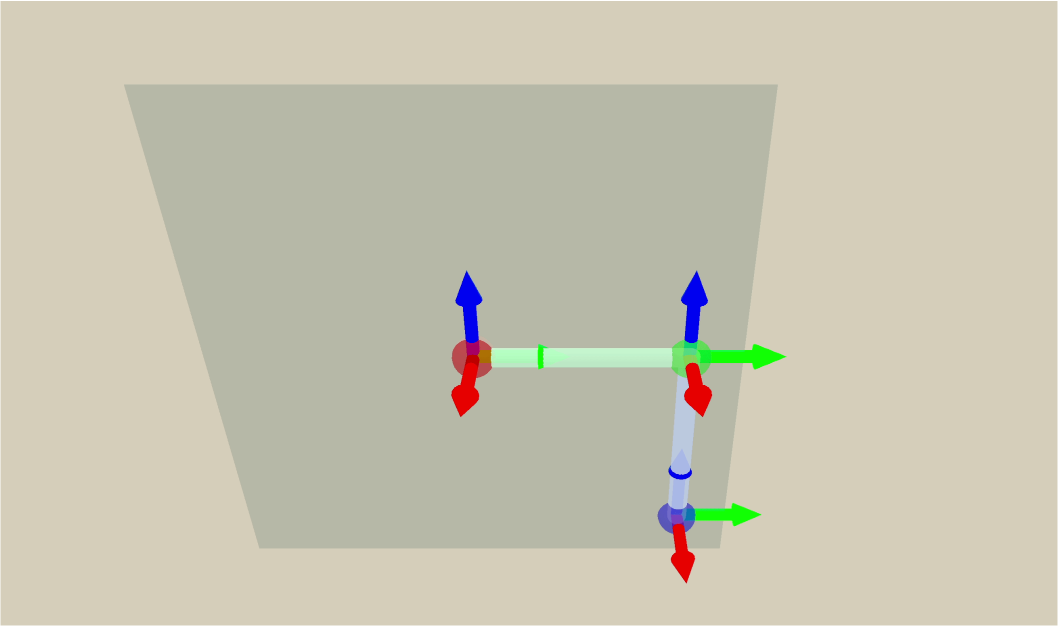
- Two observers record the same motion as $\xi_{b(t)}^{s_1}$ and $\xi_{b(t)}^{s_2}$
- **What is the relationship between $\xi_{b(t)}^{s_1}$ and $\xi_{b(t)}^{s_2}$?**

Example 1 of Change of Frame

- From our previous example, we know that the motion of the end-effector (blue sphere) can be recorded in the spatial frame (red sphere) as $\xi^s_{s\to b(t)}$:

$$[\xi^s_{s\to b(t)}] = \dot{T}^s_{s\to b(t)}(T^s_{s\to b(t)})^{-1} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\alpha & 0 \\ 0 & \alpha & 0 & -\alpha \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

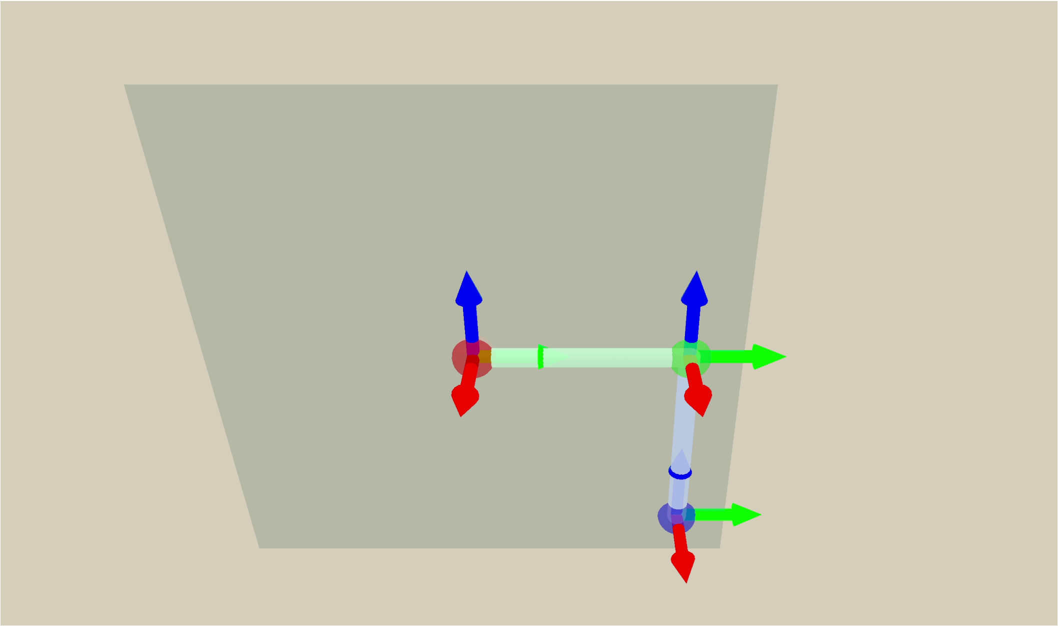
$$\xi^s_{s\to b(t)} = [0,0,-\alpha,\alpha,0,0]^T$$



Example 1 of Change of Frame

- From our previous example, we know that the motion of the end-effector (blue sphere) can be recorded in the spatial frame (red sphere) as $\xi^{s}_{s\to b(t)}$:

- Now we introduce a new frame $\mathscr{F}_o$, the frame of the green sphere. How can we record the same motion by $\mathscr{F}_o$ as $\xi^{o}_{s\to b(t)}$?



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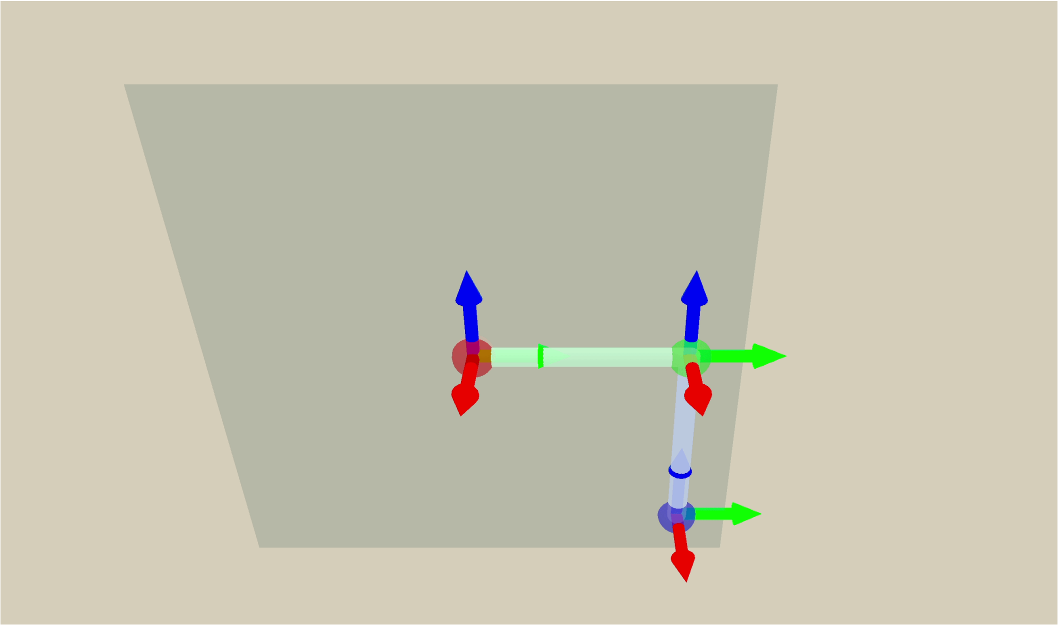
- From our previous example, we know that the motion of the end-effector (blue sphere) can be recorded in the spatial frame (red sphere) as $\xi^{s}_{s\to b(t)}$:

- By simple inspection, we can find end-effector is rotating about the x-axis of $\mathscr{F}_o$ and the instant velocity along the axis is zero

$$\omega^o = [\alpha, 0, 0]^T$$
$$\hat{\omega}^o = [1, 0, 0]^T$$
$$q^o = [0, 0, 0]^T$$
$$\mathbf{v}^o_\omega = [0, 0, 0]^T$$



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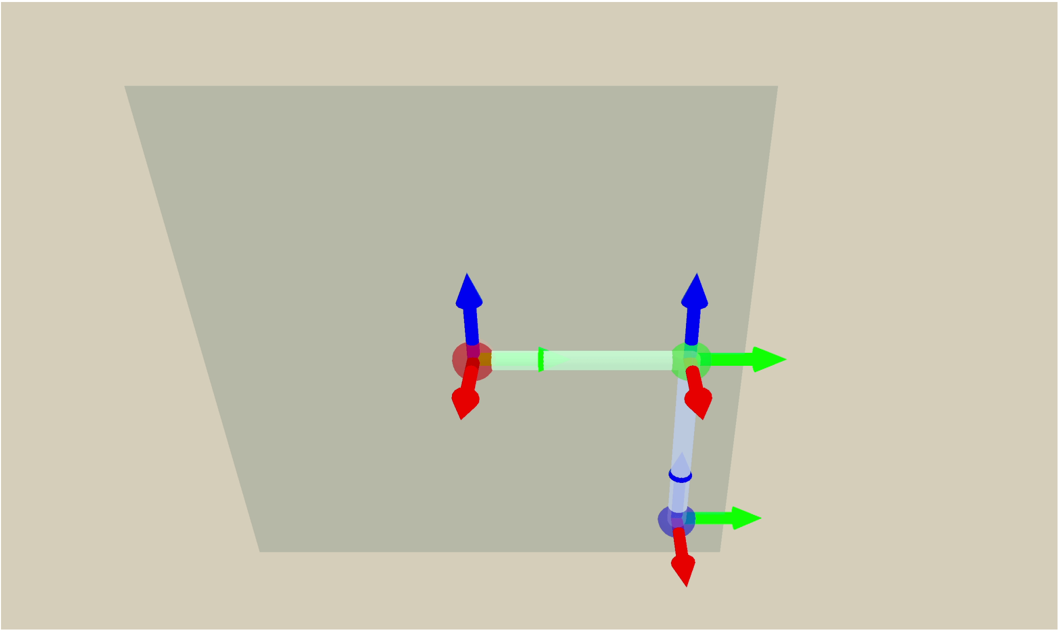
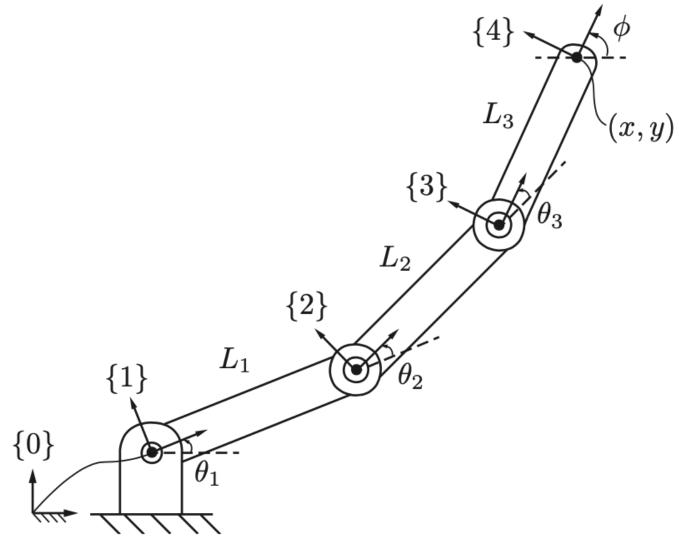
- Recall: $\xi^o = \begin{bmatrix} -[\omega^o]q^o + \mathbf{v}^o_\omega \\ \omega^o \end{bmatrix}$

- Thus we have $\xi^o_{s\to b(t)} = [0,0,0,\alpha,0,0]^T$



Example 2 of Change of Frame

- For the 3-link robot arm



- Given $\xi^3_{L_3(t)}$, what is $\xi^0_{L_3(t)}$? Assume the transformation is $T^0_{L_0 \to L_3(t)}$ at time t.

Change of Frame by Similarity Transformation

- For two observers, one records by $\mathscr{F}_{s_1}$ and the other by $\mathscr{F}_{s_2}$, then
  - $\dot{T}^{s_1}_{s'\to b(t)} = [\xi^{s_1}_{b(t)}]T^{s_1}_{s'\to b(t)}$
  - $\dot{T}^{s_2}_{s'\to b(t)} = [\xi^{s_2}_{b(t)}]T^{s_2}_{s'\to b(t)}$

Change of Frame by Similarity Transformation

$$[\xi^{s_1}_{b(t)}] = T_{s_1 \to s_2}[\xi^{s_2}_{b(t)}]T^{-1}_{s_1 \to s_2}$$

- When the observer's frame changes,
  - twist also conforms to the similarity transformation

Change of Frame by Similarity Transformation

- By $T^{s_1}_{s'\to b(t)} = T_{s_1\to s_2} T^{s_2}_{s'\to b(t)} (T_{s_1\to s_2})^{-1}$,

$$\dot{T}^{s_1}_{s'\to b(t)} = T_{s_1\to s_2}\dot{T}^{s_2}_{s'\to b(t)}(T_{s_1\to s_2})^{-1} \Leftrightarrow [\xi^{s_1}_{b(t)}]T^{s_1}_{s'\to b(t)} = T_{s_1\to s_2}[\xi^{s_2}_{b(t)}]T^{s_2}_{s'\to b(t)}(T_{s_1\to s_2})^{-1}$$

$$\Leftrightarrow [\xi^{s_1}_{b(t)}] = T_{s_1\to s_2}[\xi^{s_2}_{b(t)}]T^{s_2}_{s'\to b(t)}(T_{s_1\to s_2})^{-1}(T^{s_1}_{s'\to b(t)})^{-1}$$

$$\Leftrightarrow [\xi^{s_1}_{b(t)}] = T_{s_1\to s_2}[\xi^{s_2}_{b(t)}]T^{s_2}_{s'\to b(t)}\{(T_{s_1\to s_2})^{-1}(T^{s_1}_{s'\to b(t)})^{-1}T_{s_1\to s_2}\}(T_{s_1\to s_2})^{-1}$$

$$\Leftrightarrow [\xi^{s_1}_{b(t)}]T_{s_1\to s_2} = T_{s_1\to s_2}[\xi^{s_2}_{b(t)}]T^{b}_{s'\to b(t)}(T^{b}_{s'\to b(t)})^{-1}(T_{s_1\to s_2})^{-1}$$

$$\Leftrightarrow [\xi^{s_1}_{b(t)}]T_{s_1\to s_2} = T_{s_1\to s_2}[\xi^{s_2}_{b(t)}]$$

$$[\xi^{s_1}_{b(t)}] = T_{s_1\to s_2}[\xi^{s_2}_{b(t)}]T^{-1}_{s_1\to s_2}$$

Read by Yourself

Adjoint Matrix

$$[\xi_{b(t)}^{s_1}] = T_{s_1 \to s_2}[\xi_{b(t)}^{s_2}]T_{s_1 \to s_2}^{-1}$$

- $\xi_{b(t)}^{s_1}$ is linear w.r.t. $\xi_{b(t)}^{s_2}$
- We introduce a matrix $[\mathrm{Ad}_{T_{s_1 \to s_2}}] \in \mathbb{R}^{6 \times 6}$ to relate them:

$$\xi_{b(t)}^{s_1} = [Ad_{T_{s_1 \to s_2}}]\xi_{b(t)}^{s_2}$$

- Do computation based on the similarity transformation, and you can get

$$[\mathrm{Ad}_{T_{s_1 \to s_2}}] = \begin{bmatrix} R_{s_1 \to s_2} & [\mathbf{t}_{s_1 \to s_2}]R_{s_1 \to s_2} \\ 0 & R_{s_1 \to s_2} \end{bmatrix}$$

Spatial Twist and Body Twist

- If we observe the motion of the body
  - from $\mathscr{F}_s$, the velocity is $\xi^{s}_{b(t)}$ **(spatial twist)**
  - from the moving object $\mathscr{F}_b$, the velocity is $\xi^{b(t)}_{b(t)}$ **(body twist)**

Spatial Twist and Body Twist

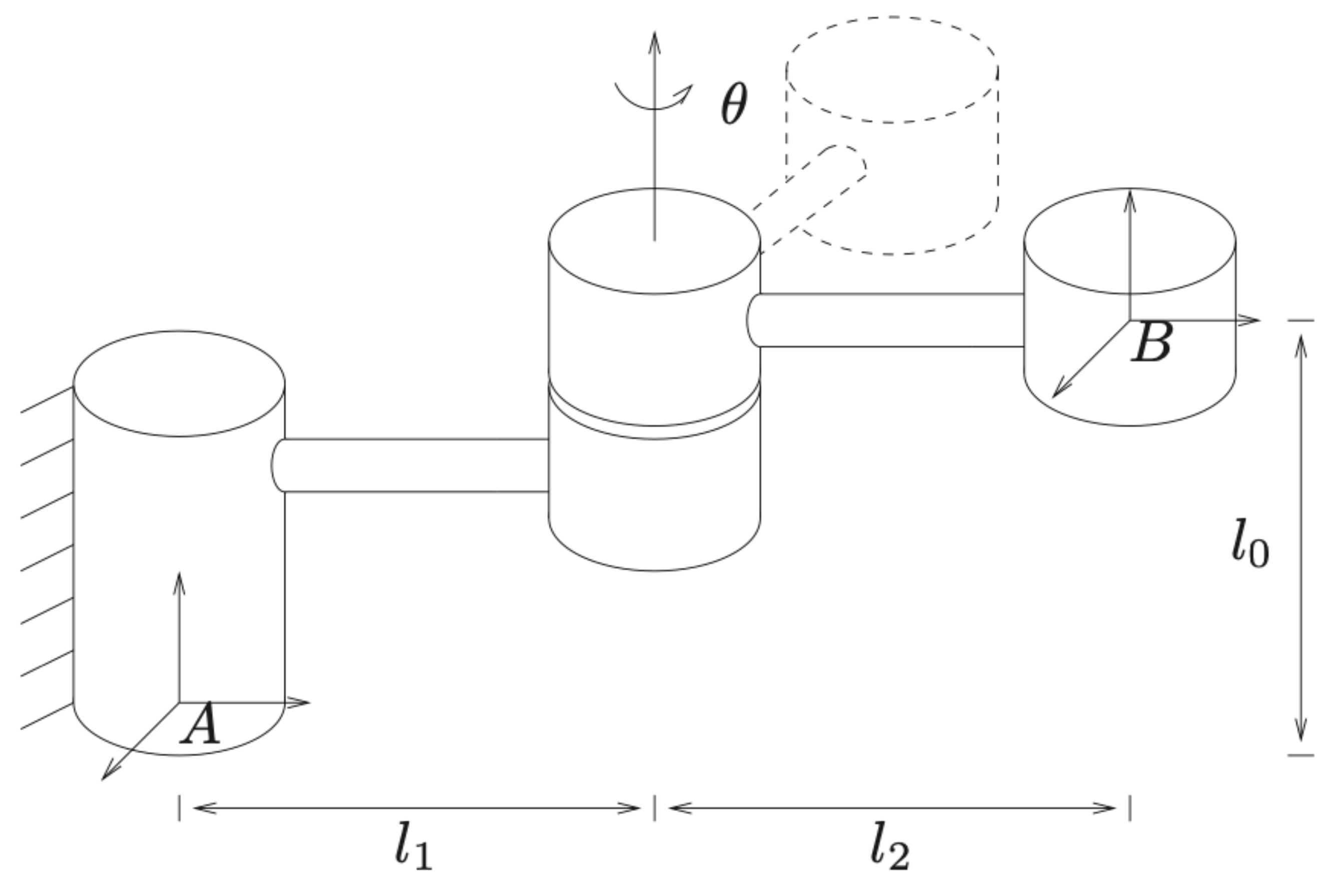
- By $\dot{T}^s_{s'\to b(t)} = [\xi^s_{b(t)}]T_{s'\to b(t)}$, $[\xi^s_{b(t)}] = \dot{T}^s_{s'\to b(t)}(T^s_{s'\to b(t)})^{-1}$
- Note that we take $s' = s$ here
- Using the similarity transformation to change the frame, we have
  - $T^s_{s\to b(t)}[\xi^{b(t)}_{b(t)}](T_{s\to b(t)})^{-1} = \dot{T}^s_{s\to b(t)}(T^s_{s\to b(t)})^{-1}$
  - $\therefore [\xi^{b(t)}_{b(t)}] = (T_{s\to b(t)})^{-1}\dot{T}^s_{s\to b(t)}$

Example 3 of Change of Frame

- Given the motion of rigid-body

$$T_{A\to B(t)} = \begin{bmatrix} \cos\theta(t) & -\sin\theta(t) & 0 & -l_2\sin\theta(t) \\ \sin\theta(t) & \cos\theta(t) & 0 & l_1 + l_2\cos\theta(t) \\ 0 & 0 & 1 & l_0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- What is the spatial twist?
- What is the body twist?



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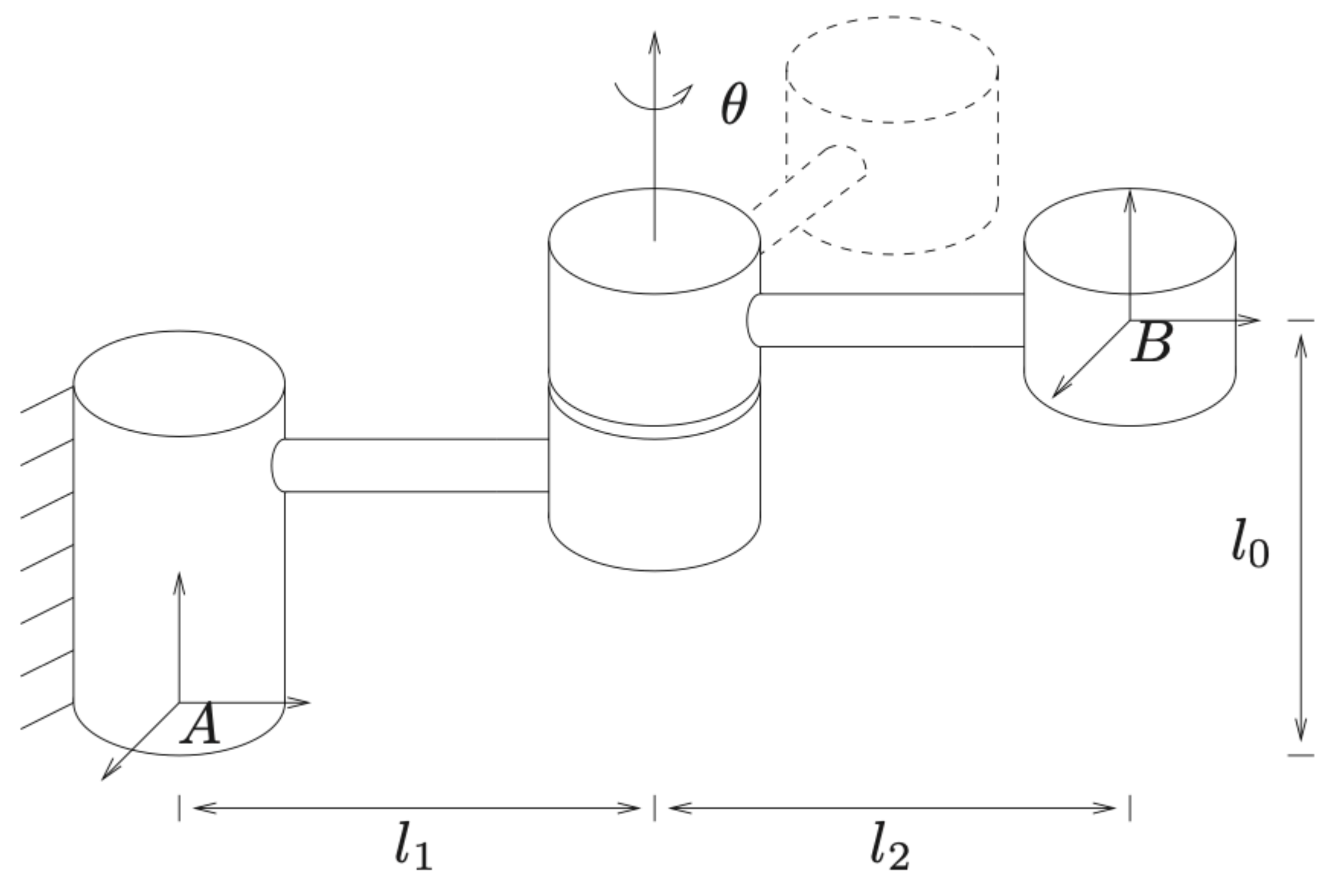
- $[\xi^A_{B(t)}] = \dot{T}_{A\to B(t)} T^{-1}_{A\to B(t)}$

$$\xi^A_{B(t)} = [l_1, 0, 0, 0, 0, 1]^T$$

- $[\xi^B_{B(t)}] = T^{-1}_{A\to B(t)} \dot{T}_{A\to B(t)}$

$$\xi^{B(t)}_{B(t)} = [-l_2, 0, 0, 0, 0, 1]^T$$



```python
import sympy as sp
from sympy import *

t = symbols("t")
l0 = symbols("l0")
l1 = symbols("l1")
l2 = symbols("l2")

T = Matrix(symarray('T', (4, 4)))
T[0, 0] = cos(t)
T[0, 1] = -sin(t)
T[0, 2] = 0
T[0, 3] = -l2 * sin(t)
T[1, 0] = sin(t)
T[1, 1] = cos(t)
T[1, 2] = 0
T[1, 3] = l1 + l2 * cos(t)
T[2, 0] = 0
T[2, 1] = 0
T[2, 2] = 1
T[2, 3] = l0
T[3, 0] = 0
T[3, 1] = 0
T[3, 2] = 0
T[3, 3] = 1

xi_s = sp.diff(T, t) @ sp.Inverse(T)
xi_s.simplify()

xi_b = sp.Inverse(T) @ sp.diff(T, t)
xi_b.simplify()
```

Example 3 of Change of Frame

- By $T_{A\to B(t)} = \begin{bmatrix} \cos\theta(t) & -\sin\theta(t) & 0 & -l_2\sin\theta(t) \\ \sin\theta(t) & \cos\theta(t) & 0 & l_1 + l_2\cos\theta(t) \\ 0 & 0 & 1 & l_0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, we have

$R_{A\to B(t)} = \begin{bmatrix} \cos\theta(t) & -\sin\theta(t) & 0 \\ \sin\theta(t) & \cos\theta(t) & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $\mathbf{t}_{A\to B(t)} = \begin{bmatrix} -l_2\sin\theta(t) \\ l_1 + l_2\cos\theta(t) \\ l_0 \end{bmatrix}$.

- By $[\mathrm{Ad}_{T_{A\to B(t)}}] = \begin{bmatrix} R_{A\to B(t)} & [\mathbf{t}_{A\to B(t)}]R_{A\to B(t)} \\ 0 & R_{A\to B(t)} \end{bmatrix}$,

$$[\mathrm{Ad}_{T_{A\to B(t)}}] = \begin{bmatrix} \cos\theta(t) & -\sin\theta(t) & 0 & -l_0\sin\theta(t) & -l_0\cos\theta(t) & l_1 + l_2\cos\theta(t) \\ \sin\theta(t) & \cos\theta(t) & 0 & l_0\cos\theta(t) & -l_0\sin\theta(t) & l_2\sin\theta(t) \\ 0 & 0 & 1 & -l_1\cos\theta(t) - l_2 & l_1\sin\theta(t) & 0 \\ 0 & 0 & 0 & \cos\theta(t) & -\sin\theta(t) & 0 \\ 0 & 0 & 0 & \sin\theta(t) & \cos\theta(t) & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Example 3 of Change of Frame

By $\xi^A_{A\to B(t)} = [l_1,0,0,0,0,1]^T$

$\xi^B_{A\to B(t)} = [-l_2,0,0,0,0,1]^T$

$$[\mathrm{Ad}_{T_{s\to b}}] = \begin{bmatrix} \cos\theta(t) & -\sin\theta(t) & 0 & -l_0\sin\theta(t) & -l_0\cos\theta(t) & l_1 + l_2\cos\theta(t) \\ \sin\theta(t) & \cos\theta(t) & 0 & l_0\cos\theta(t) & -l_0\sin\theta(t) & l_2\sin\theta(t) \\ 0 & 0 & 1 & -l_1\cos\theta(t) - l_2 & l_1\sin\theta(t) & 0 \\ 0 & 0 & 0 & \cos\theta(t) & -\sin\theta(t) & 0 \\ 0 & 0 & 0 & \sin\theta(t) & \cos\theta(t) & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

We can verify that $\xi^A_{A\to B(t)} = [\mathrm{Ad}_{T_{s\to b}}]\xi^B_{A\to B(t)}$

Summary

- Twist ξ denotes the 6D motion velocity
- Relationship with $\dot{T}$: $\dot{T}^{o}_{s'\to b(t)} = [\xi^{o}_{b(t)}]T^{o}_{s'\to b(t)}$
- Change of frame:
  - $[\xi^{s_1}_{b(t)}] = T_{s_1\to s_2}[\xi^{s_2}_{b(t)}]T^{-1}_{s_1\to s_2}$
  - $\xi^{s_1}_{b(t)} = [Ad_{T_{s_1\to s_2}}]\xi^{s_2}_{b(t)}$
- Spatial twist: $[\xi^{s}_{b(t)}] = \dot{T}^{s}_{s'\to b(t)}(T^{s}_{s'\to b(t)})^{-1}$
- Body twist: $[\xi^{b(t)}_{b(t)}] = (T_{s\to b(t)})^{-1}\dot{T}^{s}_{s\to b(t)}$