Rotation

Jiayuan Gu

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Agenda

- Robot: Multi-Body System
- Math about Rotation: $\mathbb{SO}(3)$ and $\mathbb{SE}(3)$
- Representations of Rotation
  - Axis-Angle
  - Quaternions
  - Euler Angles
- Local Structure of $\mathbb{SO}(3)$

Robot: Multi-Body System

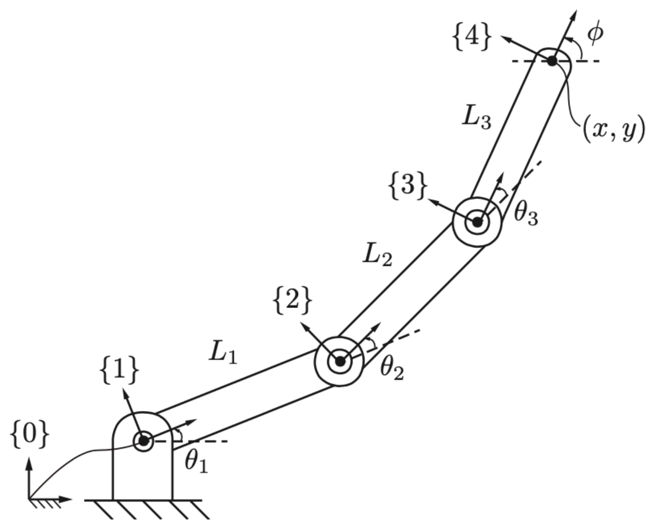
Link and Joint

Link:

- **Links** are interconnected rigid bodies
- Usually a chain (one parent)

Joint:

- **Joints** are the connectors between links. They determine the DoF of motion between adjacent links

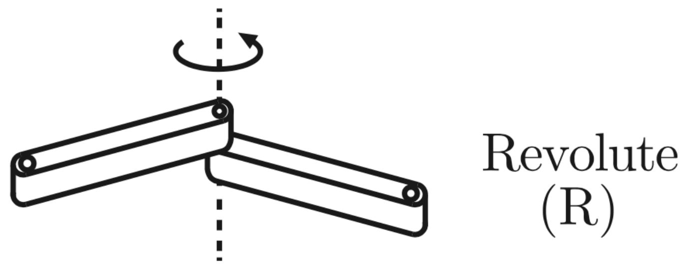


Base Link and End-Effector Link

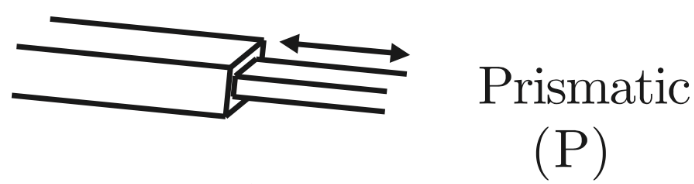
- Base (Root) link:
  - The first link
  - Regarded as the “fixed” reference
  - The spatial frame $\mathcal{F}_s$ is attached to it
- End-effector link
  - The last link (e.g., gripper)
  - A frame $\mathcal{F}_e$ is attached to it

Two Common Joint Types

- Revolute/Hinge/Rotational joint



- Prismatic/Translational joint

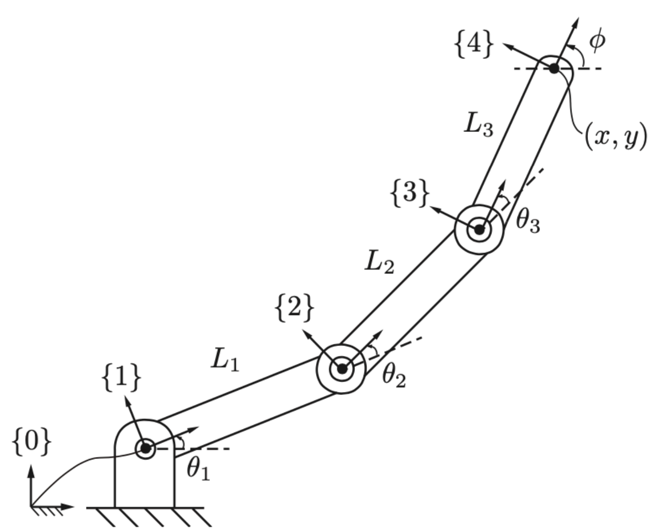


Kinematics:
The Geometry of Motion

- Kinematics: describing the **motion** of bodies (position and velocity) **without considering the forces** that cause them to move

Kinematic Configuration

- Assuming frames are assigned to each link, we can parameterize **the pose of each joint**
  - Using the relative **angle** and **translation** between adjacent frames



Kinematic Configuration

- Two representations of the end-effector pose
  - **Joint space:** The space in which each coordinate is a vector of joint poses (**angles** around **joint axis**)
  - **Cartesian space:** The space of the rigid transformations of the end-effector by $(R_{s \to e}, \mathbf{t}_{s \to e})$, where $\mathcal{F}_e$ is the end-effector frame

Kinematics Equations

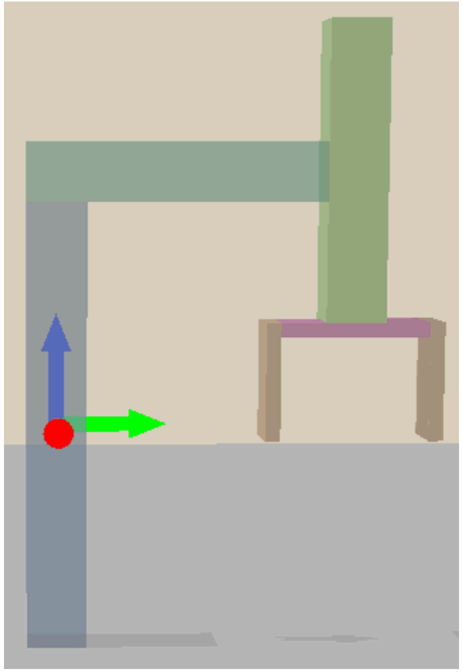
- "Define how **input movement** at one or more joints specifies the configuration of the device, in order to **achieve a task position** or end-effector location."

- Map the joint space coordinate $\theta \in \mathbb{R}^n$ to a transformation matrix T:

$$T_{s \to e} = f(\theta)$$

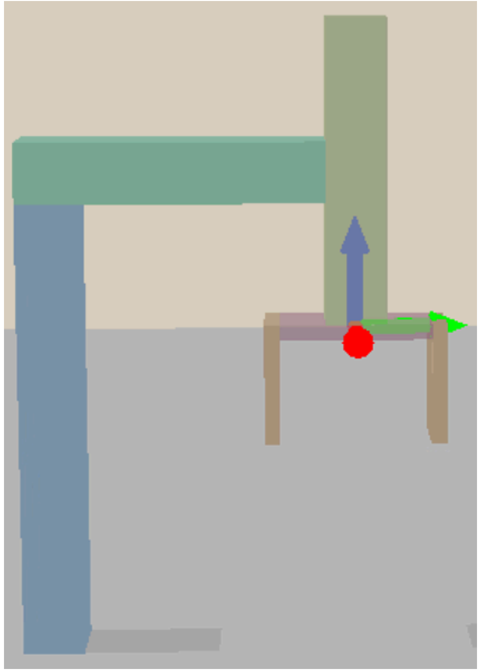
- Calculated by composing transformations along the kinematic chain

https://en.wikipedia.org/wiki/Kinematics_equations



base

$\rightarrow$



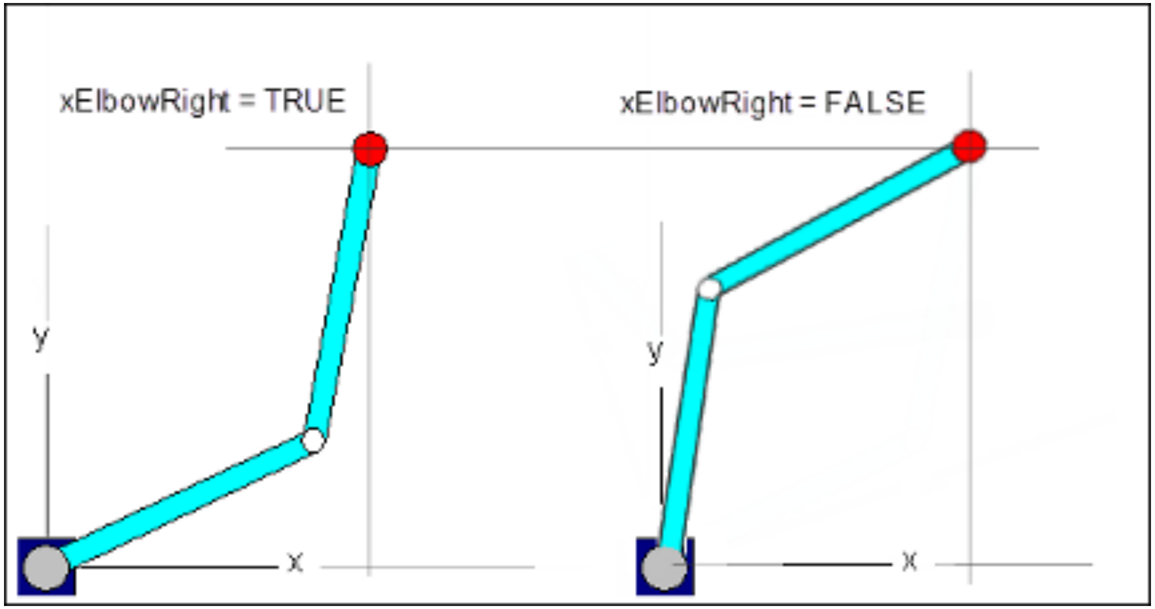
end_effector

$$T^0_{0\to 3} = T^0_{0\to 1} T^1_{1\to 2} T^2_{2\to 3} = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 & -\sin\theta_1(l_2 + l_3) \\ \sin\theta_1 & \cos\theta_1 & 0 & \cos\theta_1(l_2 + l_3) \\ 0 & 0 & 1 & l_1 - l_4 + \theta_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Inverse Kinematics

- Given the forward kinematics $T_{s\to e}(\theta)$ and the target pose $T_{target} = \mathbb{SE}(3)$, find solutions θ that satisfy $T_{s\to e}(\theta) = T_{target}$

Inverse Kinematics



Solutions may not be unique

https://product-help.schneider-electric.com/Machine%20Expert/V1.2/en/codesys_softmotion/topics/_sm_configure_kinematics.htm

How to Relate the Motion in Joint Space and Cartesian Space?

- Q1 (Forward Kinematics): If the robot moves by $\Delta\theta$ in the joint space, how much will the end-effector move in the Cartesian space?

- Q2 (Inverse Kinematics): If the robot would move the end-effector by Δx in the Cartesian space, how shall it change its joint poses?

Differentiability of Transformation

- $T_{s \to e} = f(\theta)$
  - $T_{s \to e(t)} = f(\theta)$, $T_{s \to e(t+\Delta t)} = f(\theta + \Delta\theta)$
  - $T_{s \to e(t+\Delta t)} - T_{s \to e(t)} = f(\theta + \Delta\theta) - f(\theta)$

- We will study the differentiability of rigid transformations, starting from rotations

$\mathbb{SO}(3)$ and $\mathbb{SE}(3)$

Rotation in $\mathbb{R}^3$

3 Degree of Freedoms

$\mathbb{SO}(3)$: The Space of Rotations

- $\mathbb{SO}(n) = \{R \in \mathbb{R}^{n \times n} : \det(R) = 1, RR^T = I\}$
- $\mathbb{SO}(n)$: “Special Orthogonal Group”
  - “Group”: roughly, closed under matrix multiplication
  - “Orthogonal”: $RR^T = I$
  - “Special”: $\det(R) = 1$

- Examples:
  - $\mathbb{SO}(2)$: 2D rotations, 1 DoF
  - $\mathbb{SO}(3)$: 3D rotations, 3 DoF

$\mathbb{SE}(3)$: The Space of Rigid Transformations

- $\mathbb{SE}(3) := \left\{ T = \begin{bmatrix} R & \mathbf{t} \\ 0 & 1 \end{bmatrix}, R \in \mathbb{SO}(3), \mathbf{t} \in \mathbb{R}^3 \right\}$

- $\mathbb{SE}(3)$: "Special Euclidean Group"
  - "Group": roughly, closed under matrix multiplication
  - "Euclidean": R and $\mathbf{t}$
  - "Special": $\det(R) = 1$

- $\mathbb{SE}(3)$ has 6 DoF

- We need some theoretical understanding of $\mathbb{SO}(3)$ and $\mathbb{SE}(3)$
  - Parameterization
  - Topological structure

Axis-Angle Representation

Euler's Rotation Theorem

- Any composition of rotations (in 3D space) is equivalent to a single rotation about a fixed axis $\hat{\omega} \in \mathbb{R}^3$ ($\|\hat{\omega}\| = 1$) through a positive angle θ
  - $\hat{\omega}$: unit vector of rotation axis
  - θ: angle of rotation

- It indicates that the set of rotations has a group structure
  - $R \in \mathbb{SO}(3) := \mathrm{Rot}(\hat{\omega}, \theta)$

https://en.wikipedia.org/wiki/Euler%27s_rotation_theorem

Given $\hat{\omega}$ and θ, what is $R \in \mathbb{SO}(3)$?

Skew-Symmetric Matrix

- A is skew-symmetric $A = -A^T$
- Skew-symmetric matrix operator:

$$a = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, [a] := \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}$$

- Cross product can be a linear transformation
  - $a \times b = [a]b$

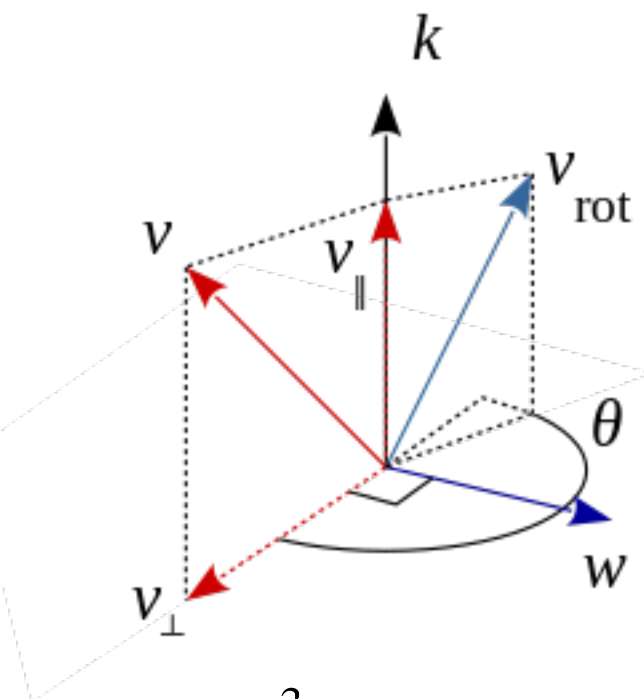
Given $\hat{\omega}$ and θ, what is $R \in \mathbb{SO}(3)$?

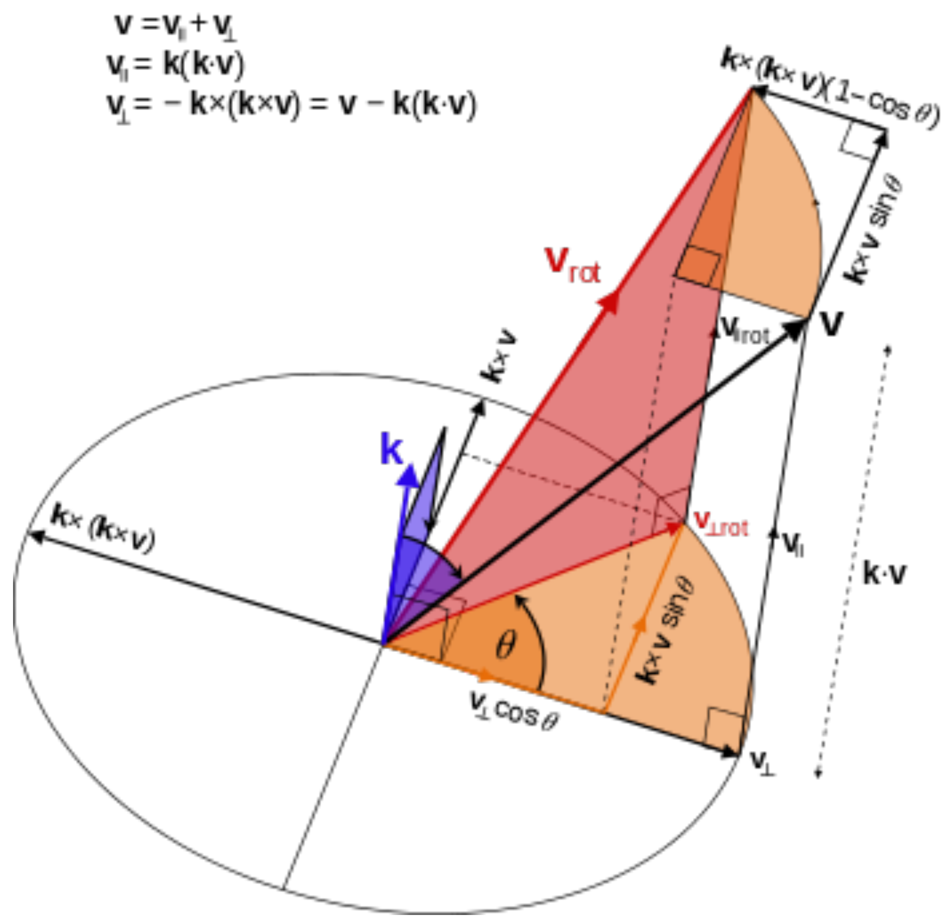
- We can show that, for any $x \in \mathbb{R}^3$

$$\begin{aligned}\mathrm{Rot}(\hat{\omega}, \theta)x &= x + (\sin\theta)\hat{\omega} \times x + (1 - \cos\theta)\hat{\omega} \times (\hat{\omega} \times x) \\ &= \{I + [\hat{\omega}]\sin\theta + [\hat{\omega}]^2(1 - \cos\theta)\}x \qquad (1)\end{aligned}$$

Derivation: https://en.wikipedia.org/wiki/Rotation_matrix

Illustration





$k \in \mathbb{R}^3$ is the rotation axis

$v \in \mathbb{R}^3$ is the vector to rotate

$v_{rot} \in \mathbb{R}^3$ is the vector rotated

Given $\hat{\omega}$ and θ, what is $R \in \mathbb{SO}(3)$?

- We can show that, for any $x \in \mathbb{R}^3$
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- By Taylor's expansion of **sin**, **cos**, $[\hat{\omega}]^3 = -[\hat{\omega}]$, and above
$$\text{Rot}(\hat{\omega}, \theta)x = (I + \theta[\hat{\omega}] + \frac{\theta^2}{2!}[\hat{\omega}]^2 + \frac{\theta^3}{3!}[\hat{\omega}]^3 + \cdots)x$$

Derivation: https://en.wikipedia.org/wiki/Rotation_matrix

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- Recall Taylor's expansion of exponential,
$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots$$

Derivation: https://en.wikipedia.org/wiki/Rotation_matrix

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- Recall Taylor's expansion of exponential,
$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots$$
- Formally, we have:
$$\mathrm{Rot}(\hat{\omega}, \theta)x = e^{[\hat{\omega}]\theta}x, \forall x \in \mathbb{R}^3$$

Derivation: https://en.wikipedia.org/wiki/Rotation_matrix

Given $\hat{\omega}$ and θ, what is $R \in \mathbb{SO}(3)$?

- By $\mathrm{Rot}(\hat{\omega}, \theta)x = e^{[\hat{\omega}]\theta}x, \forall x \in \mathbb{R}^3$,

$$\mathrm{Rot}(\hat{\omega}, \theta) \equiv e^{[\hat{\omega}]\theta}$$

- This is under such a **Definition of Matrix Exponential**:

$$e^{[\hat{\omega}]\theta} = I + \theta[\hat{\omega}] + \frac{\theta^2}{2\,!}[\hat{\omega}]^2 + \frac{\theta^3}{3\,!}[\hat{\omega}]^3 + \cdots$$

Given $\hat{\omega}$ and θ, what is $R \in \mathbb{SO}(3)$?

- In the angle-axis representation, $\mathrm{Rot}(\hat{\omega}, \theta) = e^{[\hat{\omega}]\theta}$
- $\vec{\theta} = \hat{\omega}\theta$ is also called the **rotation vector**, or **exponential coordinate**

Rodrigues Formula

- Definition of Matrix Exponential:

$$e^{[\hat{\omega}]\theta} = I + \theta[\hat{\omega}] + \frac{\theta^2}{2!}[\hat{\omega}]^2 + \frac{\theta^3}{3!}[\hat{\omega}]^3 + \cdots$$

- Sum of infinite series? **Rodrigues Formula**
  - Can prove that $[\hat{\omega}]^3 = -[\hat{\omega}]$
  - Then, use Taylor expansion of **sin** and **cos**
  - $e^{[\hat{\omega}]\theta} = I + [\hat{\omega}]\sin\theta + [\hat{\omega}]^2(1 - \cos\theta)$

Given $R \in \mathbb{SO}(3)$, what is $\hat{\omega}$ and θ?

- Is there a **unique** parametrization?

Given $R \in \mathbb{SO}(3)$, what is $\hat{\omega}$ and θ?

- Is there a **unique** parametrization? No!
    1. $(\hat{\omega}, \theta)$ and $(-\hat{\omega}, -\theta)$ give the same rotation
    2. when $R = I, \theta = 0$ and $\hat{\omega}$ can be arbitrary
    3. $(\hat{\omega}, \pi)$ and $(-\hat{\omega}, \pi)$ give the same rotation $(\mathrm{tr}(R) = -1)$

Given $R \in \mathbb{SO}(3)$, what is $\hat{\omega}$ and θ?

- If we restrict $\theta \in (0, \pi)$, a unique parameterization exists:

$$\theta = \arccos \frac{1}{2}[\mathrm{tr}(R) - 1], \quad [\hat{\omega}] = \frac{1}{2\sin\theta}(R - R^T)$$

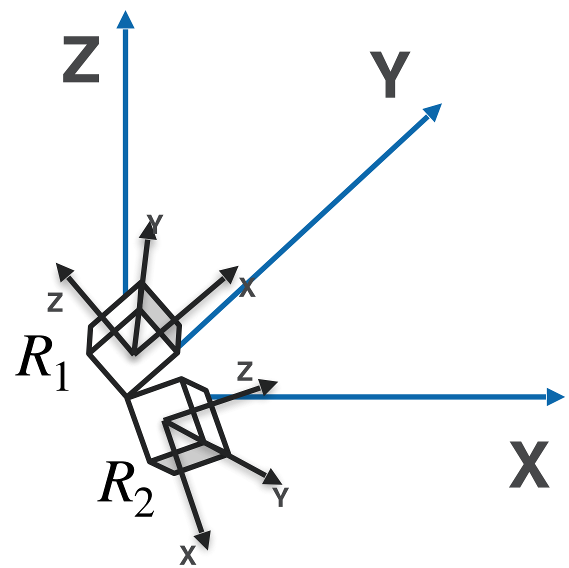
Distance between Rotations

- How to measure the distance between rotations (R_1, R_2)?
- A natural view is to measure the (minimal) effort to rotate the body at R_1 pose to R_2 pose:

$$\because (R_2 R_1^T) R_1 = R_2$$

$$\therefore \mathrm{dist}(R_1, R_2) = \theta(R_2 R_1^T)$$

$$= \arccos \frac{1}{2}[\mathrm{tr}(R_2 R_1^T) - 1]$$

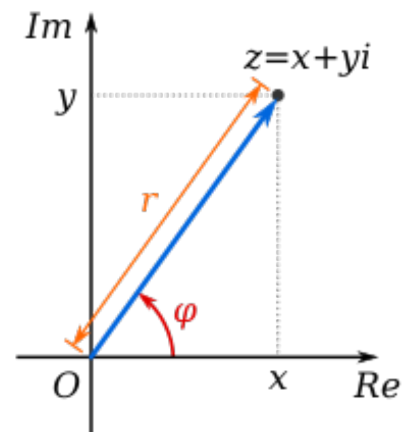


Quaternion

Note: In this section, $\vec{x} \in \mathbb{R}^3$ and $q \in \mathbb{R}^4$

Complex Number

- Recall the complex number $a + b\mathbf{i}$
  - a is the real part and $\mathbf{i}$ is the imaginary part
  - Imaginary: $\mathbf{i}^2 = -1$
  - Conjugate: $a - b\mathbf{i}$
  - Absolute value: $\sqrt{a^2 + b^2}$



https://en.wikipedia.org/wiki/Complex_number

Quaternion is a “Number”

- Recall the complex number $a + b\mathbf{i}$
- Quaternion is a more generalized complex number:

$$q = w + x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

  - w is the real part and $\vec{v} = (x, y, z)$ is the imaginary part
  - Imaginary: $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$
  - anti-commutative :
    $\mathbf{ij} = \mathbf{k} = -\mathbf{ji},\ \mathbf{jk} = \mathbf{i} = -\mathbf{kj},\ \mathbf{ki} = \mathbf{j} = -\mathbf{ik}$

Properties of General Quaternions

- Vector form: $q = (w, \vec{v})$
- Product:
  - For $q_1 = (w_1, \vec{v}_1)$ and $q_2 = (w_2, \vec{v}_2)$,
    $q_1 q_2 = (w_1 w_2 - \vec{v}_1^T \vec{v}_2, w_1 \vec{v}_2 + w_2 \vec{v}_1 + \vec{v}_1 \times \vec{v}_2)$
  - Not commutable (note that $\vec{v}_1 \times \vec{v}_2 \neq \vec{v}_2 \times \vec{v}_1$)
- Conjugate: $q^* = (w, -\vec{v})$
- Norm: $\|q\|^2 = w^2 + \vec{v}^T \vec{v} = qq^* = q^* q$
- Inverse: $q^{-1} := \dfrac{q^*}{\|q\|^2}$

Unit Quaternion as Rotation

- A **unit** quaternion $\|q\| = 1$ can represent a rotation
  - Four numbers plus one constraint $\rightarrow$ 3 DoF

- Geometrically, the shell of a 4D sphere

Build Rotation Quaternion

- Exponential coordinate → Quaternion:

$$q = [\cos(\theta/2), \sin(\theta/2)\hat{\omega}]$$

- Quaternion is very close to angle-axis representation!

- Exponential coordinate ← Quaternion:

$$\theta = 2\arccos(w), \qquad \hat{\omega} = \begin{cases} \dfrac{1}{\sin(\theta/2)}\vec{v} & \theta \neq 0 \\ 0 & \theta = 0 \end{cases}$$

Unit Quaternion as Rotation

- Rotate a vector $\vec{x}$ by a quaternion q:
  1. Augment $\vec{x}$ to $x = (0, \vec{x})$
  2. $x' = qxq^{-1}$
- Compose rotations by quaternion:
  - $(q_2(q_1xq_1^*)q_2^*)$: first rotate by q_1 and then by q_2
  - Since $(q_2(q_1xq_1^*)q_2^*) = (q_2q_1)x(q_1^*q_2^*)$, composing rotations is as simple as multiplying quaternions!

Conversation between Quaternion and Rotation Matrix

- Rotation $\leftarrow$ Quaternion

$$R(q) = E(q)G(q)^T$$

where $E(q) = [wI + [\vec{v}], -\vec{v}]$ and

$$G(q) = [wI - [\vec{v}], -\vec{v}]$$

- Rotation $\rightarrow$ Quaternion
  - Rotation $\rightarrow$ Angle-Axis $\rightarrow$ Quaternion

Double Covering

- Each rotation corresponds to two quaternions ("double-covering"): q and $-q$

More about Quaternion

- Quaternion is computationally cheap:
  - Internal representation of Physical Engine and Robot
- Pay attention to convention (w, x, y, z) or (x, y, z, w)
  - (w, x, y, z): SAPIEN, transforms3d, Eigen, blender, MuJoCo, V-Rep
  - (x, y, z, w): ROS, PhysX, PyBullet

Summary of Quaternion

- Very useful and popular in practice

- 4D parameterization, compact and efficient to compute

Euler Angles

Euler Angles are Intuitive

https://www.programmersought.com/article/8900590257/

Euler Angle to Rotation Matrix

- Rotation about principal axis is represented as:

$$R_x(\alpha) := \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix}$$

$$R_y(\beta) := \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix}$$

$$R_z(\gamma) := \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- $R = R_z(\alpha)R_y(\beta)R_x(\gamma)$ for arbitrary rotation

Euler Angles are not Unique

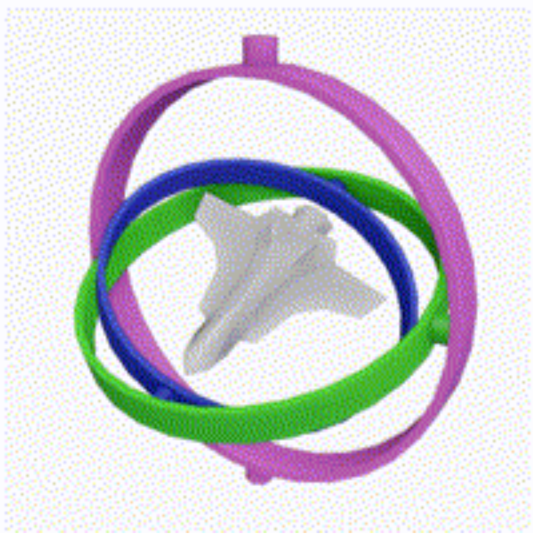
- Euler angles are not unique for some rotations.
- For example,

$$R_z(45°)R_y(90°)R_x(45°) = R_z(90°)R_y(90°)R_x(90°)$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

Gimbal Lock

- “Gimbal lock is the loss of one degree of freedom in a multi-dimensional mechanism at certain alignments of the axes.”



https://en.wikipedia.org/wiki/Gimbal_lock

Read by Yourself

Gimbal Lock

- For example: When $\beta = \pi/2$,

$$R = R_z(\alpha)R_y(\pi/2)R_x(\gamma)$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ \sin(\alpha+\gamma) & \cos(\alpha+\gamma) & 0 \\ -\cos(\alpha+\gamma) & \sin(\alpha+\gamma) & 0 \end{bmatrix}$$

since changing α and γ has the same effects, a degree of freedom disappears!

https://www.mecademic.com/resources/Euler-angles/Euler-angles

Read by Yourself

Summary of Euler Angles

- Euler angles can parameterize every rotation and has good interpretability

- It is not a unique representation at some points

- There are some points where not every change in the target space (rotations) can be realized by a change in the source space (Euler angles)

Summary of Rotation Representations

	Inverse?	Composing?
Rotation Matrix	✔	✔
Euler Angle	Complicated	Complicated
Angle-axis	✔	Complicated
Quaternion	✔	✔

Libraries

- Python
  - spicy.spatial.transform.Rotation
  - transforms3d
  - pytransform3d
- Machine learning (PyTorch): kornia, pypose
- C/C++: Eigen, ceres

Challenges in Parameterizing $\mathbb{SO}(3)$

Parameterization

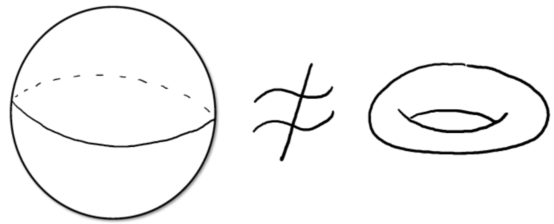
- $R \in \mathbb{SO}(3)$ is recorded by real numbers
- Parameterization is the mapping between $\mathbb{R}^d$ and $\mathbb{SO}(3)$:

$$f(\theta) = R_\theta$$

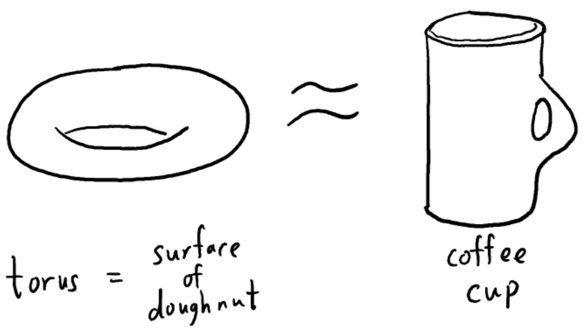
Read by Yourself

Prerequisite: Topology

- Topology: Structural Properties of a Manifold



- Two surfaces M and N are *topologically equivalent* if there is a **differentiable bijection** between M and N



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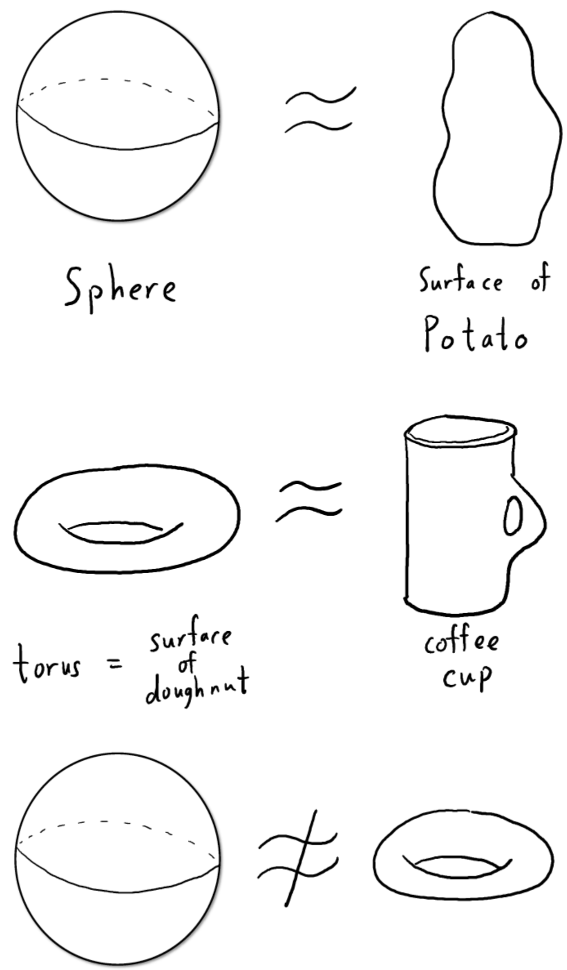
Prerequisite: Topology

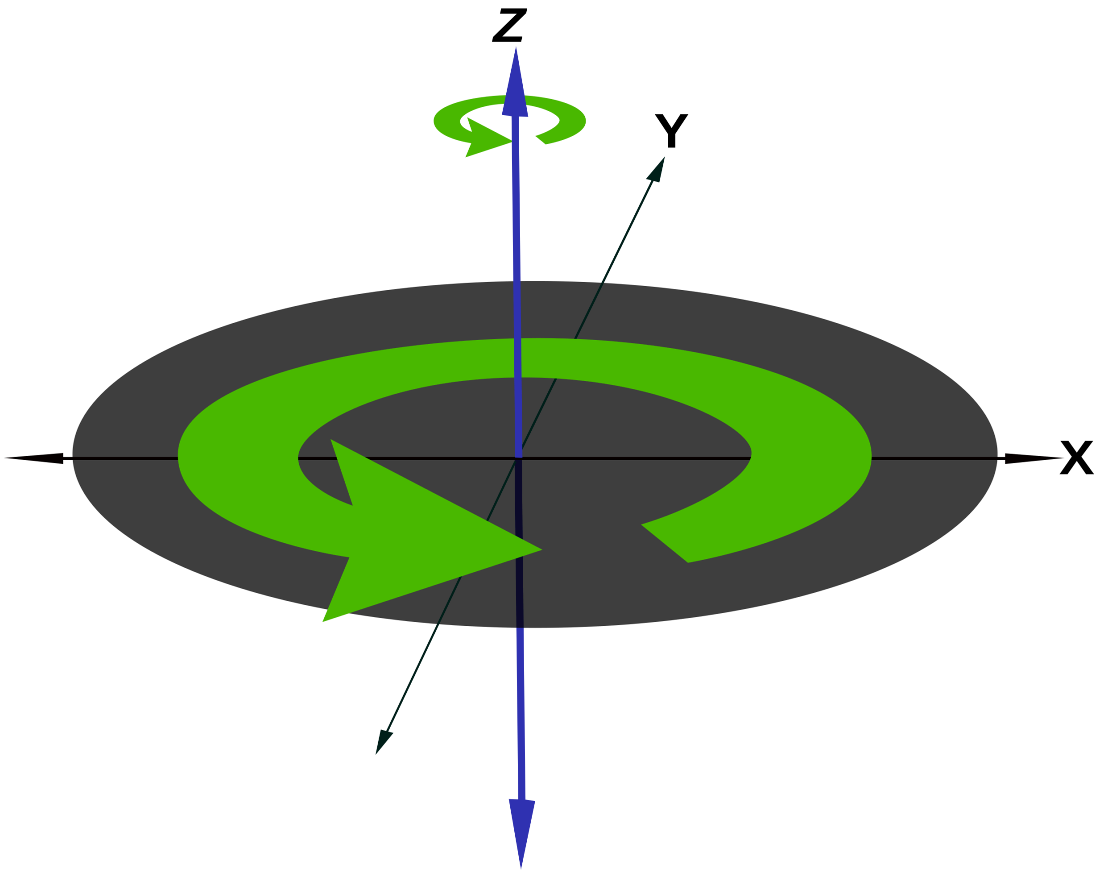
- More examples:



Read by Yourself

Topology of $\mathbb{SO}(n)$

- The topology of $\mathbb{SO}(2)$ is the same as a circle

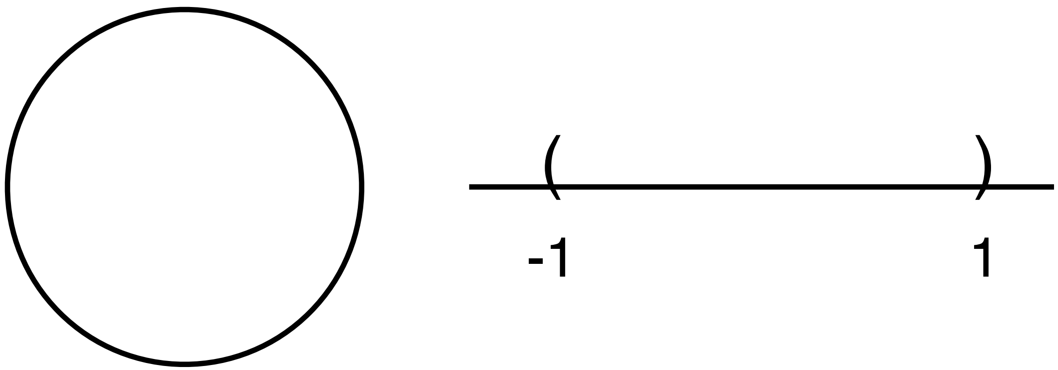


Read by Yourself

Topology of $\mathbb{SO}(n)$

- Circles do not have the same topology as $(-1,1)^n$

$\Longrightarrow$ No differentiable bijections between $\mathbb{SO}(2)$ and $(-1,1)^n$



- The topology of $\mathbb{SO}(3)$ is also different from $(-1,1)^n$

Read by Yourself

Parameterizing Rotations is Tricky

- Although $\mathbb{SO}(3)$ only has 3 DoF, you cannot build a differentiable bijection between $\mathbb{SO}(3)$ and any subset of $\mathbb{R}^3$
- Even parameterizing $\mathbb{SO}(3)$ by $\mathbb{R}^d$ with $d > 3$,
  - we cannot build differentiable bijections with $(-1,1)^d$
  - we have to either introduce constraints, or bear with singularities and the "multi-cover" issue
- The challenge brings a lot of trouble to optimization and learning

Read by Yourself

Rotation Parameterization for Neural Networks

- Zhou, Yi, et al. "On the continuity of rotation representations in neural networks." Proceedings of the IEEE/CVF conference on computer vision and pattern recognition. 2019.

- Xiang, Sitao, and Hao Li. "Revisiting the continuity of rotation representations in neural networks." arXiv preprint arXiv:2006.06234 (2020).

Read by Yourself

Local Structure of $\mathbb{SO}(3)$

Local Structure of $\mathbb{SO}(3)$

- Definition of Matrix Exponential:

$$e^{[\hat{\omega}]\theta} = I + \theta[\hat{\omega}] + \frac{\theta^2}{2!}[\hat{\omega}]^2 + \frac{\theta^3}{3!}[\hat{\omega}]^3 + \cdots$$

- Note:
  - $e^{A+B} = e^A e^B$ only when $AB - BA = 0$
- When $\theta \approx 0$, $e^{[\hat{\omega}]\theta} = I + \theta[\hat{\omega}] + o(\theta[\hat{\omega}])$

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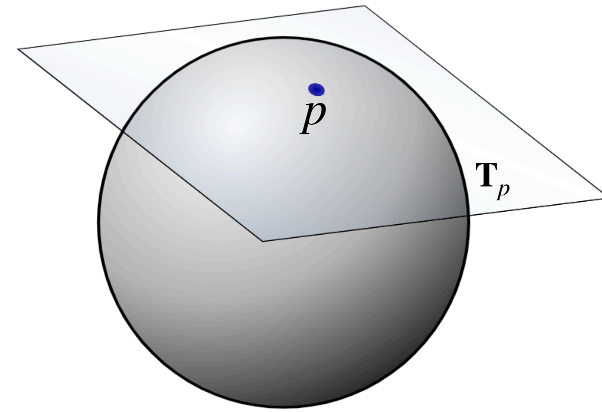
Local Structure of $\mathbb{SO}(3)$

- By $e^{[\hat{\omega}]\theta} = I + \theta[\hat{\omega}] + o(\theta[\hat{\omega}])$ when $\theta \approx 0$,

$$e^{[\vec{\theta}]} - I = [\vec{\theta}] + o([\vec{\theta}])$$



- Interpretation:
  - $[\vec{\theta}]$ forms linear subspace of $\mathbb{R}^{3\times 3}$
  - $e^{[\vec{\theta}]} \to I$ as $[\vec{\theta}] \to 0$
  - **Any** local movement in $\mathbb{SO}(3)$ around I, which is $\approx e^{[\vec{\theta}]} - I$, can be approximated by $[\vec{\theta}]$
- The set of $[\vec{\theta}]$ forms the tangent space of $\mathbb{SO}(3)$ at I

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Lie algebra $\mathfrak{so}(3)$ of $\mathbb{SO}(3)$

- The set of $[\vec{\theta}]$ is the tangent space of $\mathbb{SO}(3)$ at $R = I$
  - Ex: What is the tangent space at any $R \in \mathbb{SO}(3)$?
    - $\because e^{[\vec{\theta}]} - I = [\vec{\theta}] + o([\vec{\theta}]),\ \therefore e^{[\vec{\theta}]}R - R = [\vec{\theta}]R + o([\vec{\theta}])$
    - i.e., $\forall R' \in \mathbb{SO}(3)$ near R, $\exists \vec{\theta} \in \mathbb{R}^3$ such that $R' \approx R + [\vec{\theta}]R$
    - So the tangent space at R is $\{SR : S \in \mathbb{R}^{3\times 3}, S^T = -S\}$
- We give this set a name, the "Lie algebra of $\mathbb{SO}(3)$"
  - $\mathfrak{so}(3) := \{S \in \mathbb{R}^{3\times 3} : S^T = -S\}$

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Why called “algebra”?

- Introducing Lie bracket $[A, B] = AB - BA$, and the set of skew-symmetric matrices are closed under this binary operator
- Then, the set of skew-symmetric matrices form an “algebra”, because
  - The set is closed under Lie bracket
  - Left and right distributive law are satisfied under Lie bracket

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Tutorial for Lie Algebra

- Sola, Joan, Jeremie Deray, and Dinesh Atchuthan. "A micro Lie theory for state estimation in robotics." *arXiv preprint arXiv:1812.01537* (2018).

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