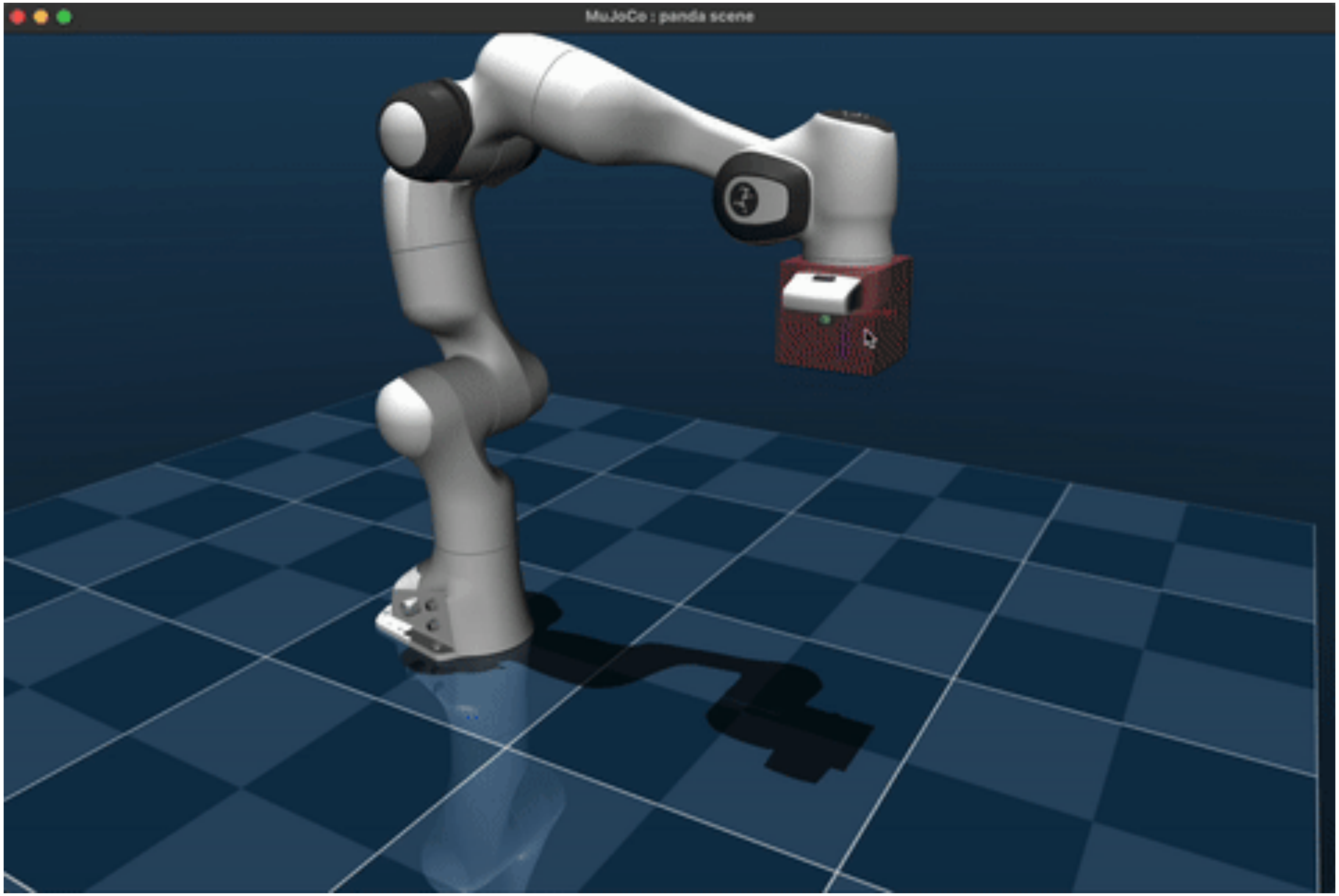
Rigid Transformation

Jiayuan Gu

Slides prepared by Prof. Hao Su with the help of
Yuzhe Qin, Minghua Liu, Fanbo Xiang, Jiayuan Gu

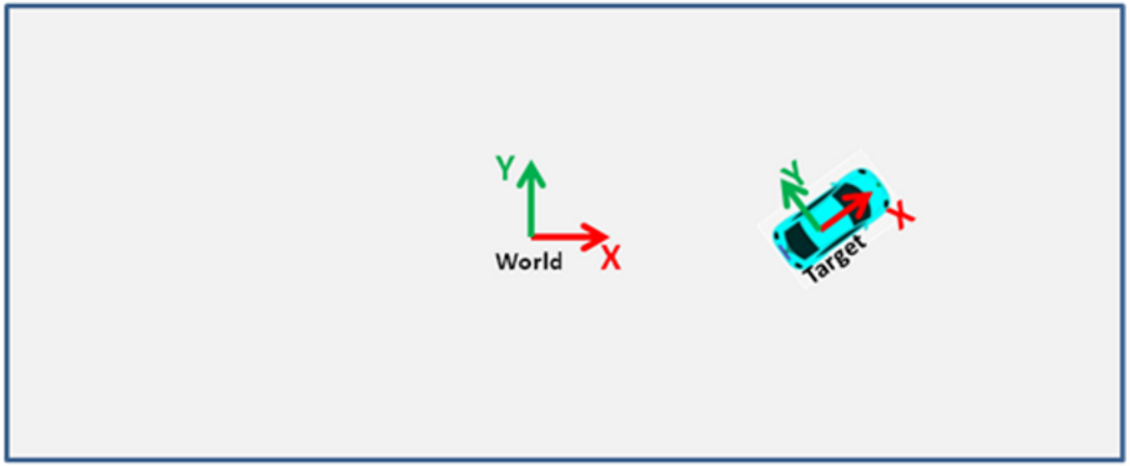
Example: End-Effector Control



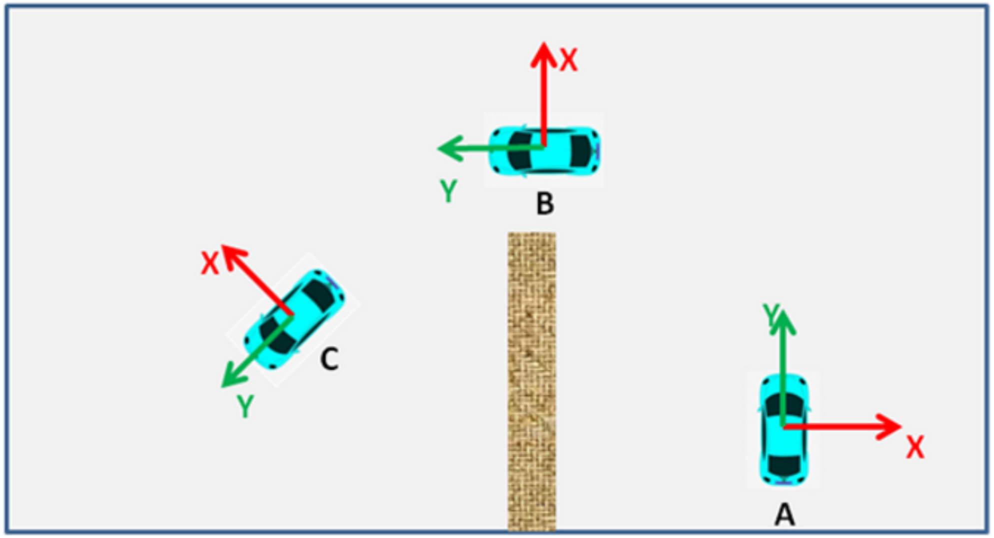
A robot can be controlled by specifying its end-effector pose

https://github.com/kevinzakka/mjctrl

Pose: Transformation between Frames



Where is the car in the world?

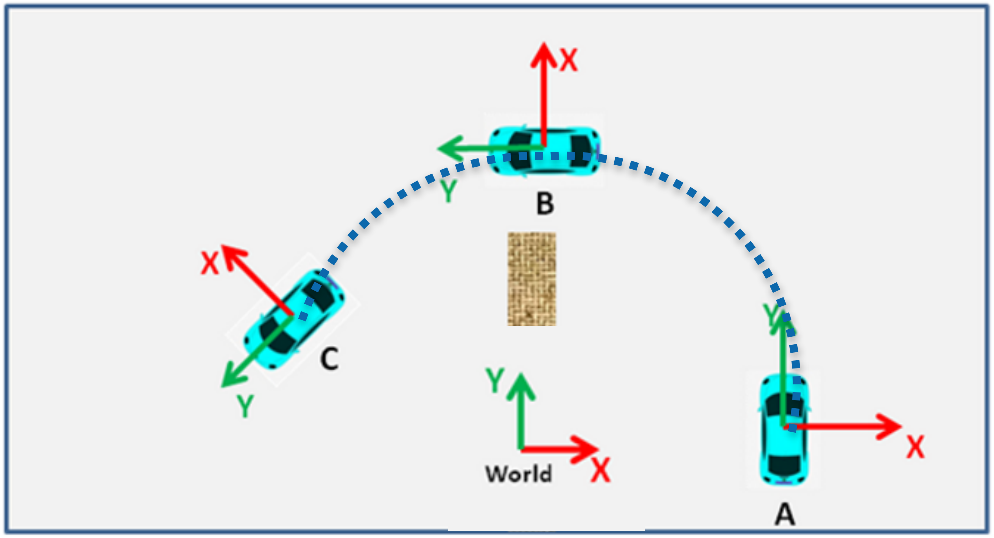


Where is the car B observed by the driver of the car A?

Describe Rigid-Body Motion

What's the motion of the car when you observe at A?

What's the motion of the car when you observe at the world frame?



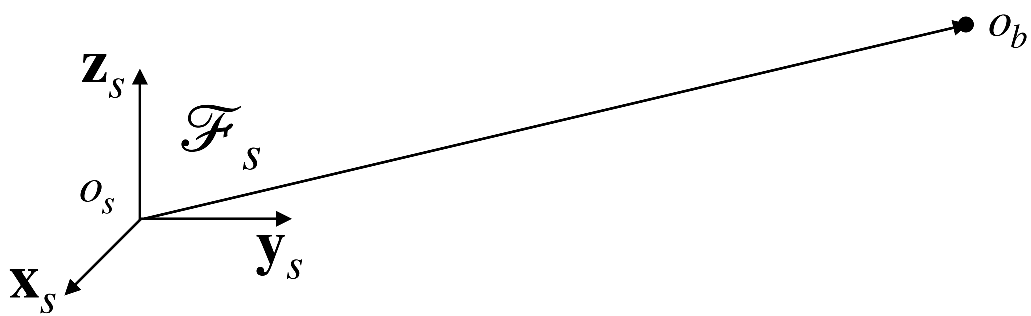
https://www.rosroboticslearning.com/rigid-body-transformations

Agenda

- Rigid Transformation
- Rigid Transformation as Linear Transformation
- Rigid Transformation for Coordinate Transformation

Rigid Transformation

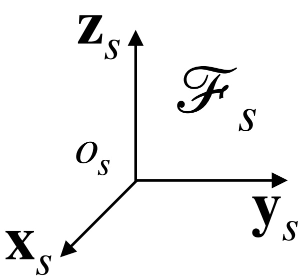
Notation Convention

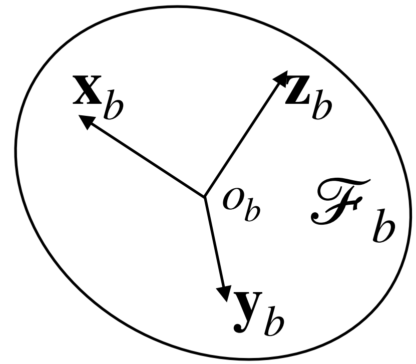


- An observer **records** the position of any point in the space **using a frame** $\mathcal{F}_s$
- We use ordinary letters to denote points (e.g., p), and bold letters to dente **vectors (e.g., $\mathbf{v}$)**
- When **writing equations**, we add a superscript to symbols to denote the recording frame, e.g.,

$$o_b^s = o_s^s + \mathbf{t}_{s \to b}^s$$

Frames

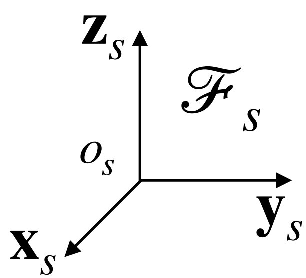


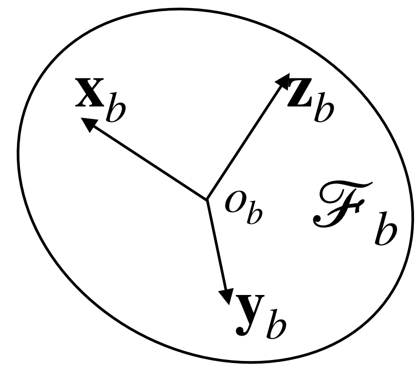


- We attach a frame $\mathscr{F}_b$ (body frame) tightly to a rigid body of interest, and $\mathscr{F}_b$ moves along with the object

- $\mathscr{F}_s$ is usually a static frame (spatial frame)

Rigid Transformation (Pose)

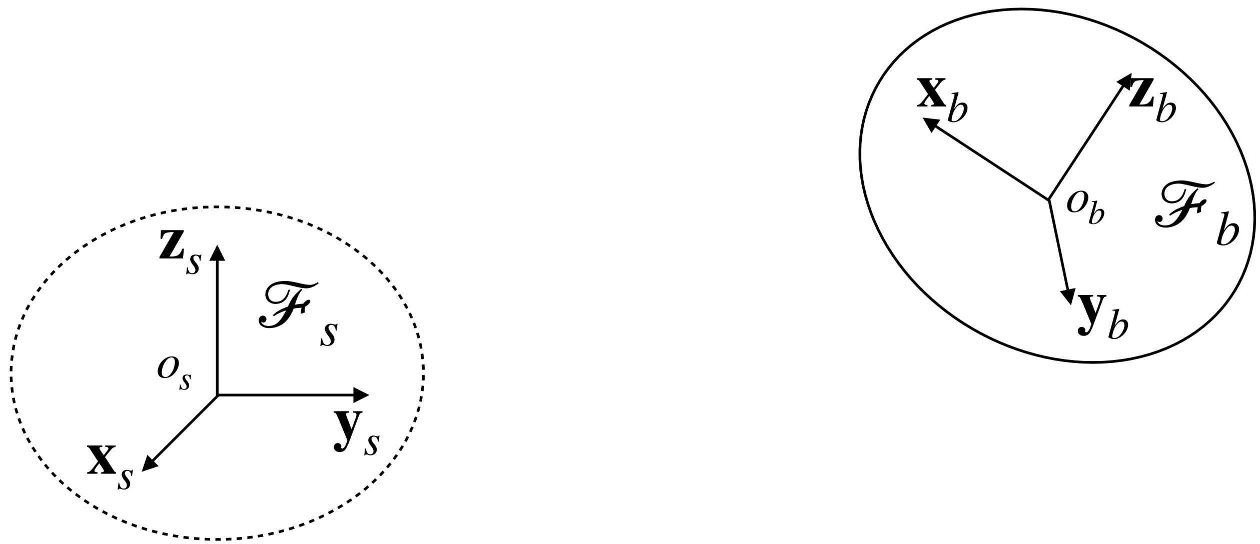




- The pose of the *rigid* object relative to $\mathscr{F}_s$:

  How to **transform** $\mathscr{F}_s$ so that it overlaps with $\mathscr{F}_b$?

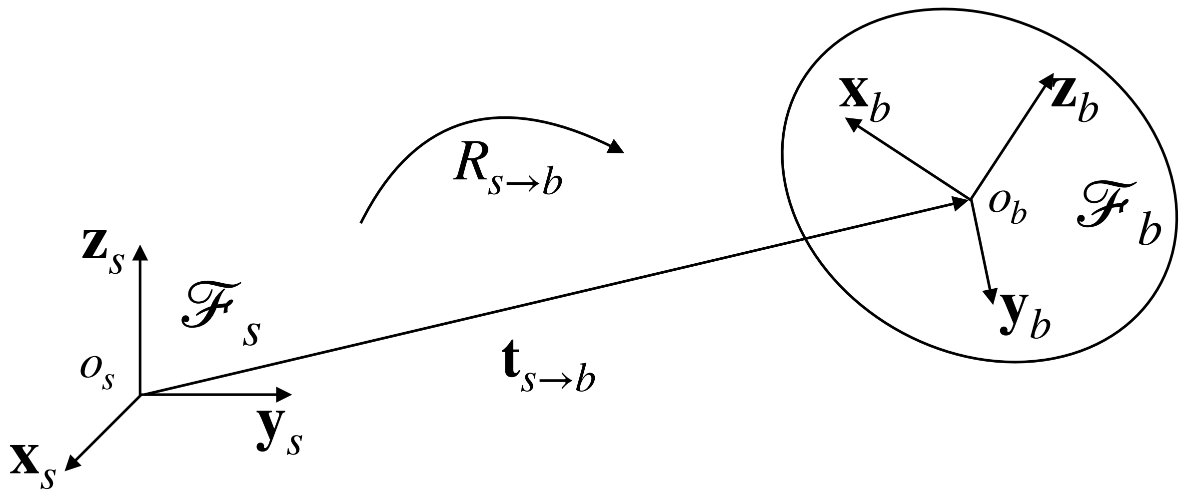
Rigid Transformation (Motion)



- The motion of the *rigid* object from $\mathscr{F}_s$ ($\mathscr{F}_{b(0)}$) to $\mathscr{F}_b$ ($\mathscr{F}_{b(t)}$)

How to **transform** $\mathscr{F}_s$ so that it overlaps with $\mathscr{F}_b$?

Rigid Transformation



- We first translate $\mathscr{F}_s$ by $\mathbf{t}_{s\to b}$ to align o_s and o_b
- And then rotate by $R_{s\to b}$ to align $\{\mathbf{x}_s, \mathbf{y}_s, \mathbf{z}_s\}$ and $\{\mathbf{x}_b, \mathbf{y}_b, \mathbf{z}_b\}$

Rigid Transformation

- "In mathematics, a **rigid transformation** is a geometric transformation of a Euclidean space that **preserves the Euclidean distance between every pair of points.**"

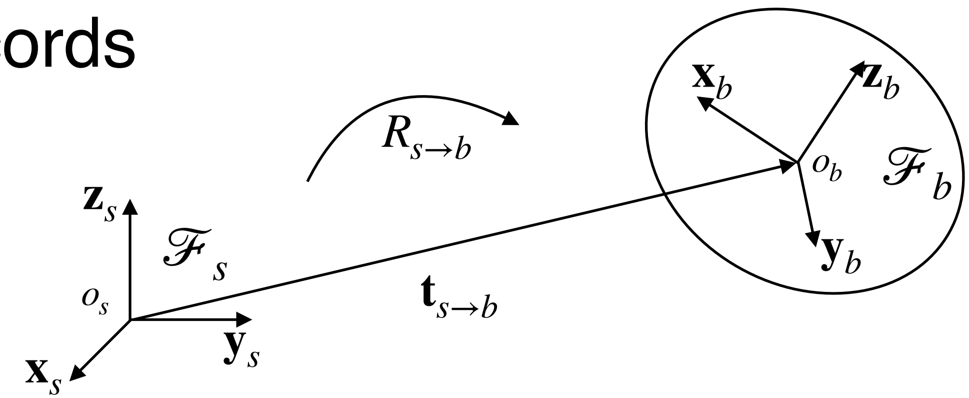
- "The rigid transformations include **rotations, translations**, reflections, or any sequence of these."

- "To avoid ambiguity, a transformation that **preserves handedness** is known as a **rigid motion**, a Euclidean motion, or a proper rigid transformation."

https://en.wikipedia.org/wiki/Rigid_transformation

Describe Rigid Transformation

- We have,
  - $o_b^s = o_s^s + \mathbf{t}_{s\to b}^s$
  - $[\mathbf{x}_b^s, \mathbf{y}_b^s, \mathbf{z}_b^s] = R_{s\to b}^s[\mathbf{x}_s^s, \mathbf{y}_s^s, \mathbf{z}_s^s]$
- Since the observer records everything using $\mathcal{F}_s$,
  - $o_s^s = 0$
  - $[\mathbf{x}_s^s, \mathbf{y}_s^s, \mathbf{z}_s^s] = I_{3\times 3}$



- Therefore,
  - $\mathbf{t}_{s\to b}^s = o_b^s$
  - $R_{s\to b}^s = [\mathbf{x}_b^s, \mathbf{y}_b^s, \mathbf{z}_b^s] \in \mathbb{R}^{3\times 3}$

$(R_{s \to b}, \mathbf{t}_{s \to b})$ as a Linear Transformation

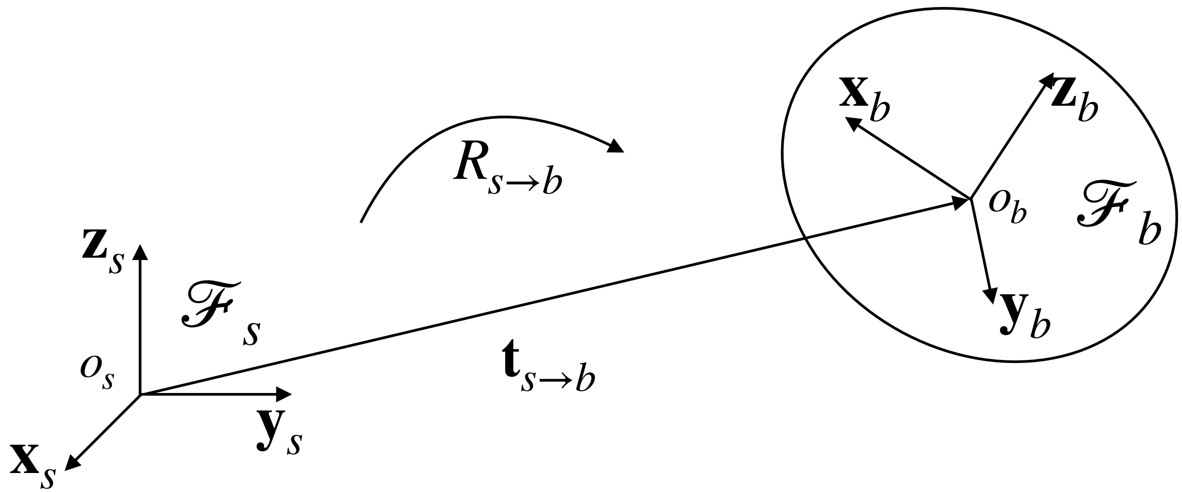
Linear Transformation

- A linear transformation of a vector space, $L : \mathbf{R}^n \rightarrow \mathbf{R}^n$, preserves linear combinations, $L(\mathbf{V}) = L(a\mathbf{v} + b\mathbf{w}) = aL(\mathbf{v}) + bL(\mathbf{w})$

- A linear transformation is a rigid transformation if it satisfies the condition, $d(L(\mathbf{v}), L(\mathbf{w})) = d(\mathbf{v}, \mathbf{w})$

https://en.wikipedia.org/wiki/Rigid_transformation

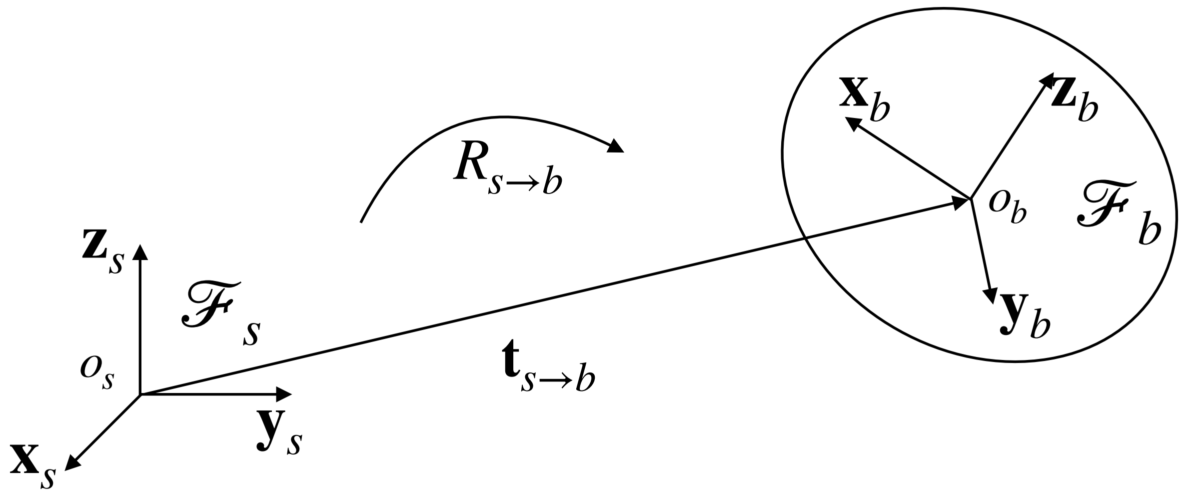
$(R_{s\to b}, \mathbf{t}_{s\to b})$ as a Linear Transformation



- $(R_{s\to b}, \mathbf{t}_{s\to b})$ transforms any **point** in the *whole space* by the following equation:

$$x'^s = R^s_{s\to b} x^s + \mathbf{t}^s_{s\to b}$$

$(R_{s \to b}, \mathbf{t}_{s \to b})$ as a Linear Transformation



- $(R_{s \to b}, \mathbf{t}_{s \to b})$ also transforms any **frame (origin + basis vectors)**

$(R_{s\to b}, \mathbf{t}_{s\to b})$ as a Linear Transformation

- Suppose $\mathcal{F}_p^s = \{p^s, (\mathbf{x}_p^s, \mathbf{y}_p^s, \mathbf{z}_p^s)\}$ is a frame at an arbitrary point p^s
- **Then, the new origin is:** $p'^s = R_{s\to b}^s p^s + \mathbf{t}_{s\to b}^s$

$(R_{s\to b}, \mathbf{t}_{s\to b})$ as a Linear Transformation

- Suppose $\mathcal{F}_p^s = \{p^s, (\mathbf{x}_p^s, \mathbf{y}_p^s, \mathbf{z}_p^s)\}$ is a frame at an arbitrary point p^s
- **Then, the new origin is:** $p'^s = R_{s\to b}^s p^s + \mathbf{t}_{s\to b}^s$
- How about the basis vectors of the frame?
  - Let's consider 3 points $\{p^s + \mathbf{x}_p^s, p^s + \mathbf{y}_p^s, p^s + \mathbf{z}_p^s\}$
  - Then, a new basis (e.g., $\mathbf{x}_p^s$) after transformation are:

$(R_{s\to b}, \mathbf{t}_{s\to b})$ as a Linear Transformation

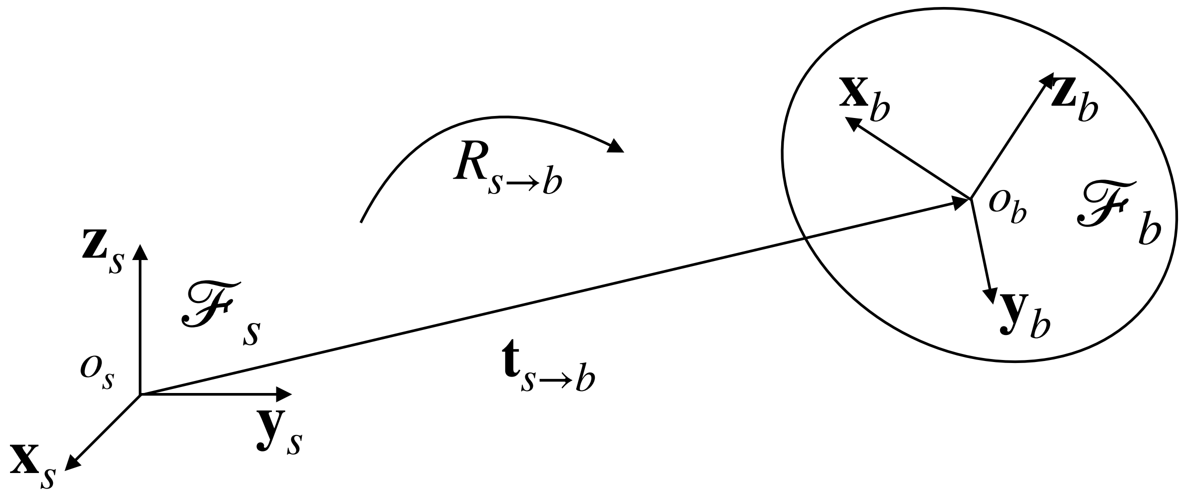
- Suppose $\mathscr{F}_p^s = \{p^s, (\mathbf{x}_p^s, \mathbf{y}_p^s, \mathbf{z}_p^s)\}$ is a frame at an arbitrary point p^s
- **Then, the new origin is:** $p'^s = R_{s\to b}^s p^s + \mathbf{t}_{s\to b}^s$
- How about the basis vectors of the frame?
  - Let's consider 3 points $\{p^s + \mathbf{x}_p^s, p^s + \mathbf{y}_p^s, p^s + \mathbf{z}_p^s\}$
  - Then, a new basis (e.g., $\mathbf{x}_p^s$) after transformation are:

$$\mathbf{x}_{p'}^s = R_{s\to b}^s(p^s + \mathbf{x}_p^s) + t_{s\to b}^s - p'^s = R_{s\to b}^s \mathbf{x}_p^s$$

- **So the new frame is:** $\mathscr{F}_{p'}^s = \{p'^s, R_{s\to b}^s[\mathbf{x}_p^s, \mathbf{y}_p^s, \mathbf{z}_p^s]\}$

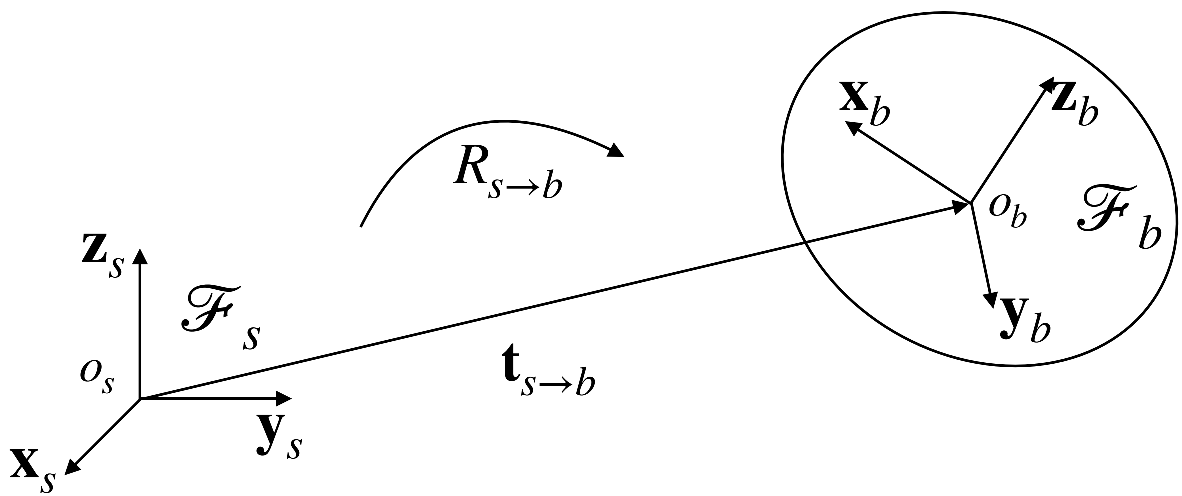
$(R_{s \to b}, \mathbf{t}_{s \to b})$ for Coordinate Transformation

Coordinate Transformation



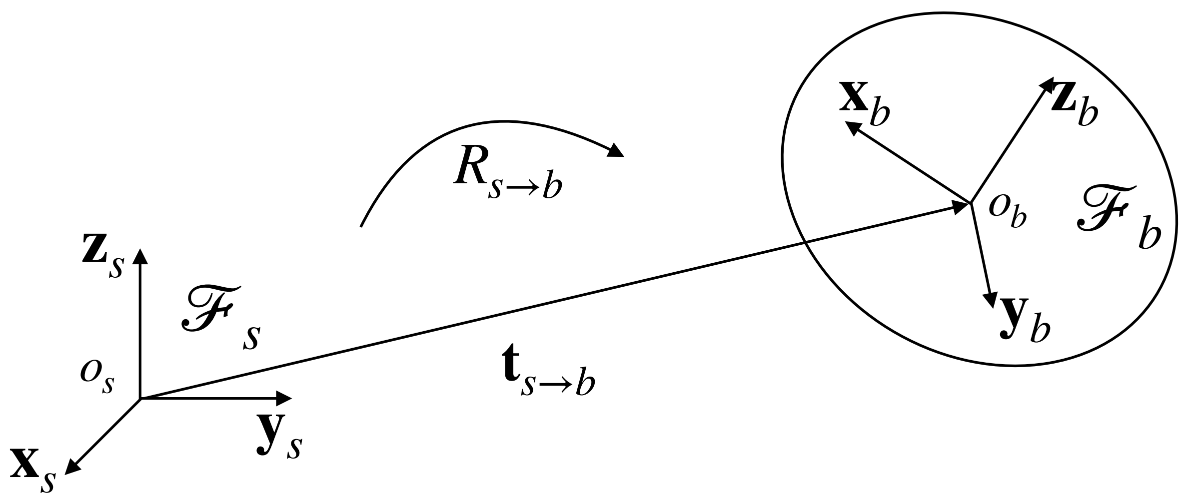
- There is a point p^b observed in the body frame $\mathscr{F}_b$, we want to know its position in the spatial frame $\mathscr{F}_s$

$(R_{s\to b}, \mathbf{t}_{s\to b})$ for Coordinate Transformation



- Imagine a process: $\mathcal{F}_b$ moves from $\mathcal{F}_s$ ($\mathcal{F}_{b(0)}$) to the current location $\mathcal{F}_{b(t)}$ within a period t.
- This is how we define $(R^s_{s\to b}, \mathbf{t}^s_{s\to b})$.

$(R_{s\to b}, \mathbf{t}_{s\to b})$ for Coordinate Transformation



- Since p moves along with $\mathcal{F}_b$ during [0, t]

$$p_t^s = R_{s\to b}^s p_0^s + \mathbf{t}_{s\to b}^s$$

- Note that $p_0^s = p_t^b$, therefore:

$$p_t^s = R_{s\to b}^s p_t^b + \mathbf{t}_{s\to b}^s$$

Homogenous Coordinates

- Homogeneous coordinate for 3D Space:

$$\tilde{x} := \begin{bmatrix} x \\ 1 \end{bmatrix} \in \mathbb{R}^4$$

- Homogeneous transformation matrix:

$$T^s_{s \to b} = \begin{bmatrix} R^s_{s \to b} & \mathbf{t}^s_{s \to b} \\ 0 & 1 \end{bmatrix}$$

- Coordinate transformation under linear form:

$$\tilde{x}^s = T^s_{s \to b} \tilde{x}^b$$

- Ignore ~ for simplicity in the future.

Homogenous Coordinates

- The coordinate transformation works for any choice of $\mathscr{F}_s$ and $\mathscr{F}_b$
- As a general rule, we have:

$$x^1 = T^1_{1\to 2} x^2$$

Some Rules of Homogenous Coordinate Transformation

By $x^1 = T^1_{1\to 2}x^2$, we have $x^2 = T^2_{2\to 1}x^1$ and $x^3 = T^3_{3\to 2}x^2$.

Therefore, $x^3 = T^3_{3\to 2}T^2_{2\to 1}x^1$. But $x^3 = T^3_{3\to 1}x^1$

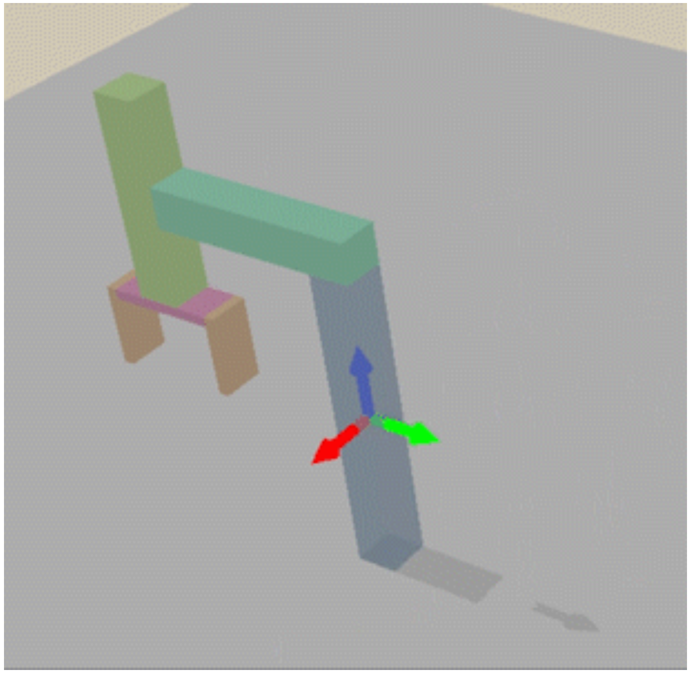
- Composition rule: $T^3_{3\to 1} = T^3_{3\to 2}T^2_{2\to 1}$

By $x^1 = T^1_{1\to 2}x^2$, we have $x^2 = (T^1_{1\to 2})^{-1}x^1$

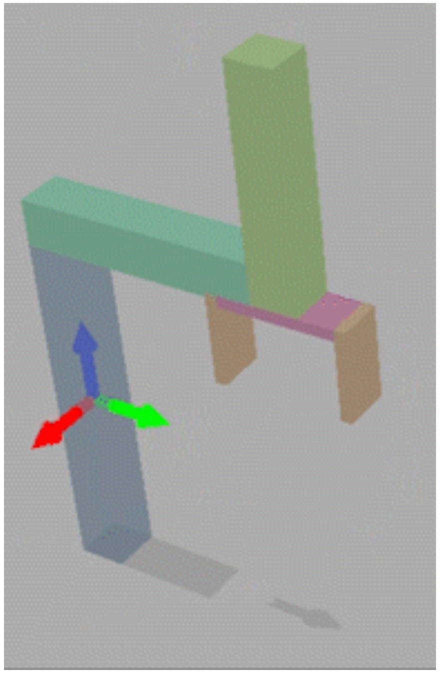
- Change of observer's frame: $T^2_{2\to 1} = (T^1_{1\to 2})^{-1}$

Example

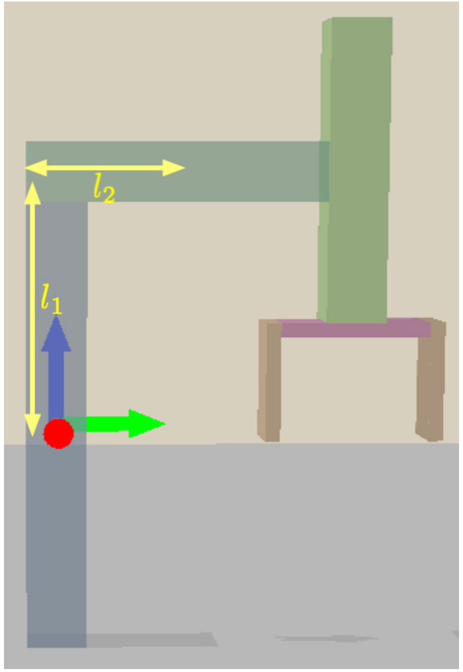
A simple 2 DoF robot arm



revolute (θ_1)

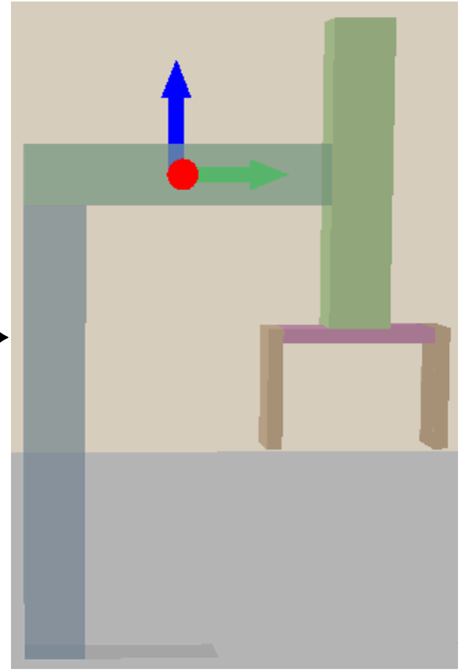


prismatic (θ_2)

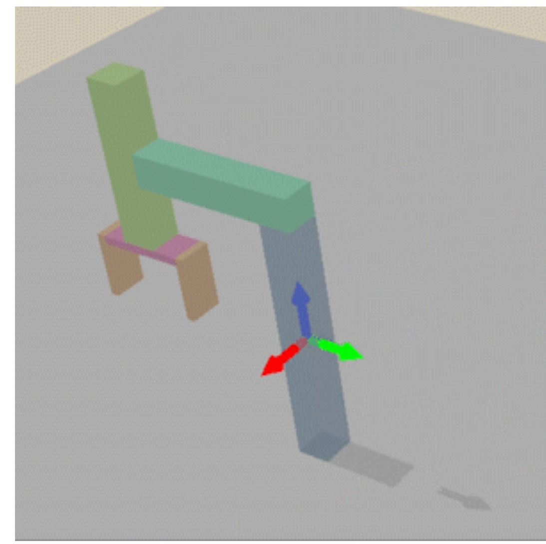


base

→

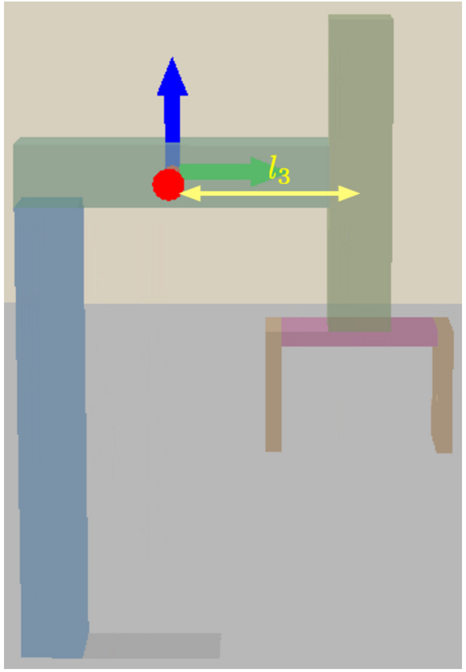


link1



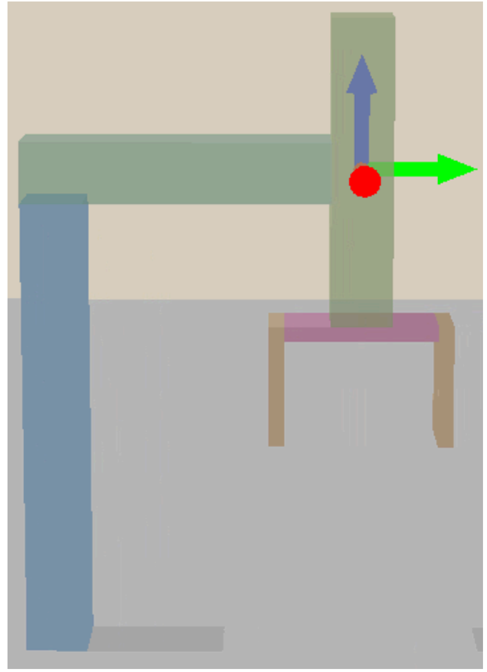
revolute (θ_1)

$$T^0_{0\to 1} = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 & -l_2\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 & 0 & l_2\cos\theta_1 \\ 0 & 0 & 1 & l_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

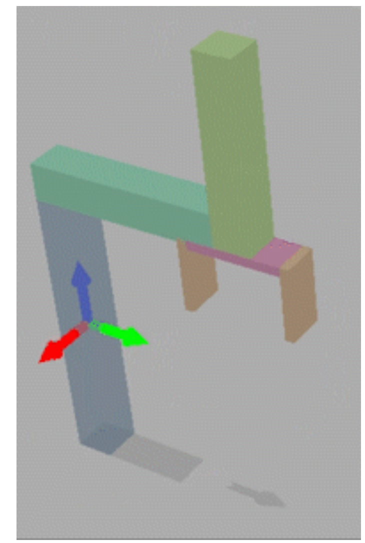


link1

→

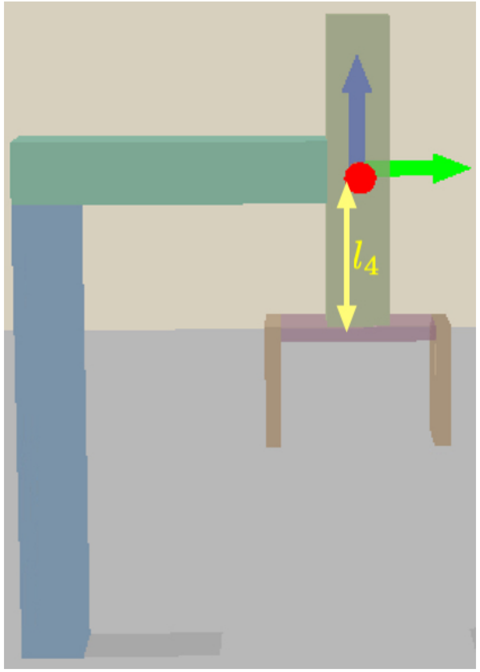


link2



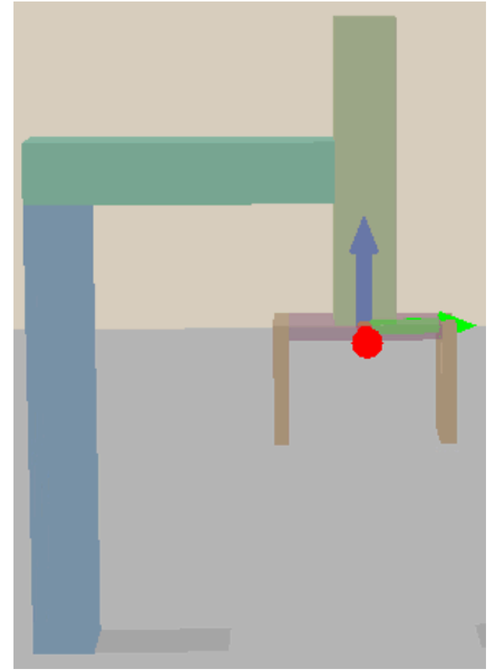
prismatic (θ_2)

$$T^1_{1\to 2} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & l_3 \\ 0 & 0 & 1 & \theta_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

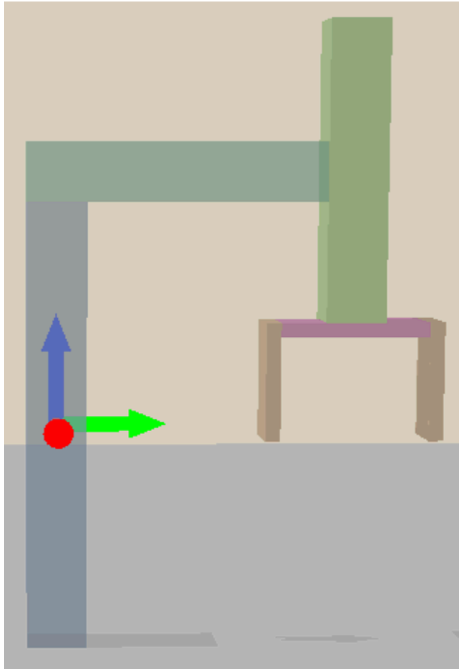


link2

→

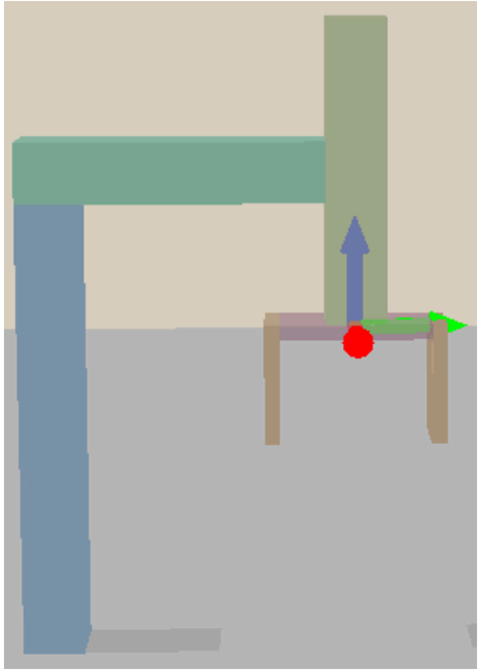

end_effector

$$T^2_{2\to3} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -l_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



base

→



end_effector

$$T^0_{0\to 3} = T^0_{0\to 1} T^1_{1\to 2} T^2_{2\to 3} = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 & -\sin\theta_1(l_2 + l_3) \\ \sin\theta_1 & \cos\theta_1 & 0 & \cos\theta_1(l_2 + l_3) \\ 0 & 0 & 1 & l_1 - l_4 + \theta_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$T^s_{1\to 2}$

- We have introduced the notations when the observer is recording via $\mathcal{F}_s$ or $\mathcal{F}_b$
  - $T^s_{s\to b}$ (record the frame alignment from $\mathcal{F}_s$ to $\mathcal{F}_b$)
  - By the change of observer's frame, we introduced $T^b_{b\to s} = (T^s_{s\to b})^{-1}$
- Next, we define the notion of $T^s_{1\to 2}$, which is how we **record** an arbitrary transformation from $\mathcal{F}_1$ to $\mathcal{F}_2$ in $\mathcal{F}_s$

Composition as a Homogeneous Linear Transformation

- The composition rule is intuitive from the perspective of linear transformation:

$$T^s_{1\to 2} = T^s_{3\to 2} T^s_{1\to 3}$$

- Try to prove:

$$T^s_{1\to 2} = T^s_{s\to 2} T^1_{1\to s}$$

Change Observer's Frame with Similarity Transformation

- Given $T^s_{1\to 2}$, what is $T^b_{1\to 2}$?

$$T^s_{1\to 2} T^s_{s\to 1} = T^s_{s\to 2}$$ Composition as Linear Transformation

$$T^s_{1\to 2} T^s_{s\to b} T^b_{b\to 1} = T^s_{s\to b} T^b_{b\to 2}$$ Composition as Coordinate Transformation

$$T^s_{1\to 2} T^s_{s\to b} T^b_{b\to 1} = T^s_{s\to b} T^b_{1\to 2} T^b_{b\to 1}$$ Composition as Linear Transformation

$$T^s_{1\to 2} T^s_{s\to b} = T^s_{s\to b} T^b_{1\to 2}$$

$$T^s_{1\to 2} = T^s_{s\to b} T^b_{1\to 2} (T^s_{s\to b})^{-1}$$

- Similarity Transformation changes the **superscript**

$B = X^{-1}AX$: Similarity Transformation

A Special Case

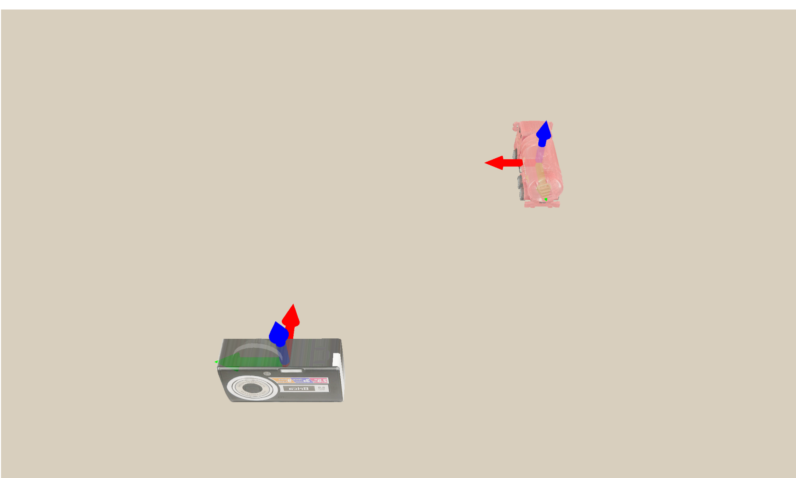
- By $T^s_{1\to 2} = T^s_{s\to b} T^b_{1\to 2} (T^s_{s\to b})^{-1}$,
  - If $\mathscr{F}_1 = \mathscr{F}_s$ and $\mathscr{F}_2 = \mathscr{F}_b$, $T^s_{s\to b} = T^b_{s\to b}$!
- Therefore, we often see the abbreviated notations:
  - $T^s_b \equiv T^s_{s\to b}$
  - $T_{s\to b} \equiv T^s_{s\to b}$
  - $T_b \equiv T^s_{s\to b}$
- The above equation can therefore be written as:

$$T^s_{1\to 2} = T_{s\to b} T^b_{1\to 2} (T_{s\to b})^{-1}$$

Example

- Consider a camera with frame $\mathscr{F}_c$ observing a red car
- Denote the current frame of the red car as $\mathscr{F}_1$

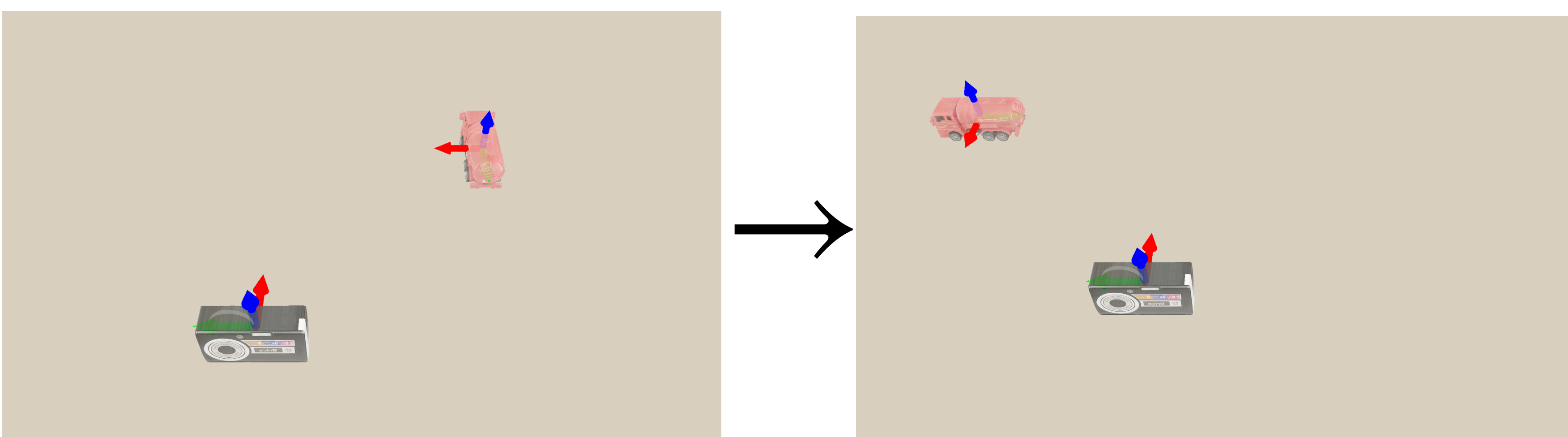
$$T^c_{c\to 1} = \begin{bmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} & 0 & l \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} & 0 & -l \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Example

- Then the red car move to a new frame $\mathscr{F}_2$

$$T^{c}_{c\to 2} = \begin{bmatrix} \cos\pi & -\sin\pi & 0 & l \\ \sin\pi & \cos\pi & 0 & l \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

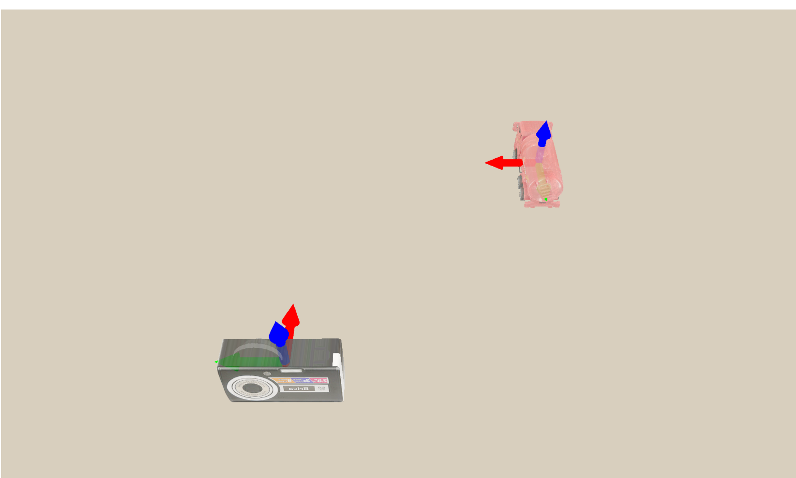


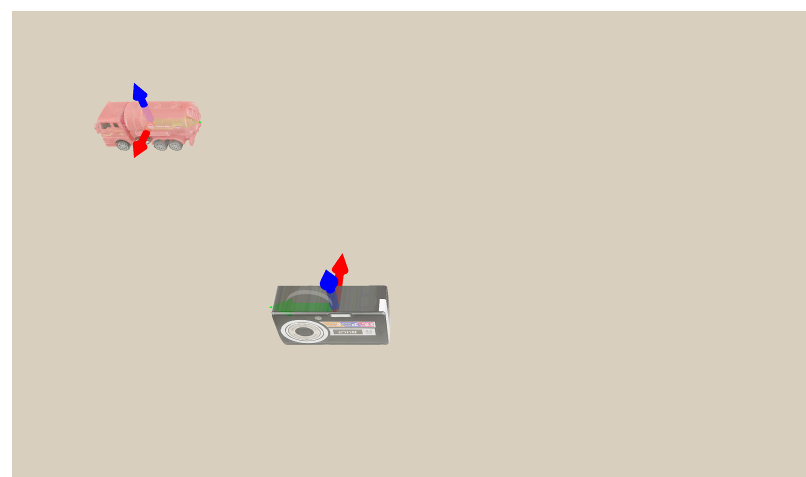
Example

- By the composition rule of coordinate transformation:

$$T^{c}_{c\to 2} = T^{c}_{c\to 1} T^{1}_{1\to 2}$$

$$T^{1}_{1\to 2} = (T^{c}_{c\to 1})^{-1} T^{c}_{c\to 2} = \begin{bmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} & 0 & 2l \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$





Example

- By the composition rule of coordinate transformation: $T^c_{c\to 2} = T^c_{c\to 1} T^1_{1\to 2}$

$$T^1_{1\to 2} = (T^c_{c\to 1})^{-1} T^c_{c\to 2} = \begin{bmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} & 0 & 2l \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

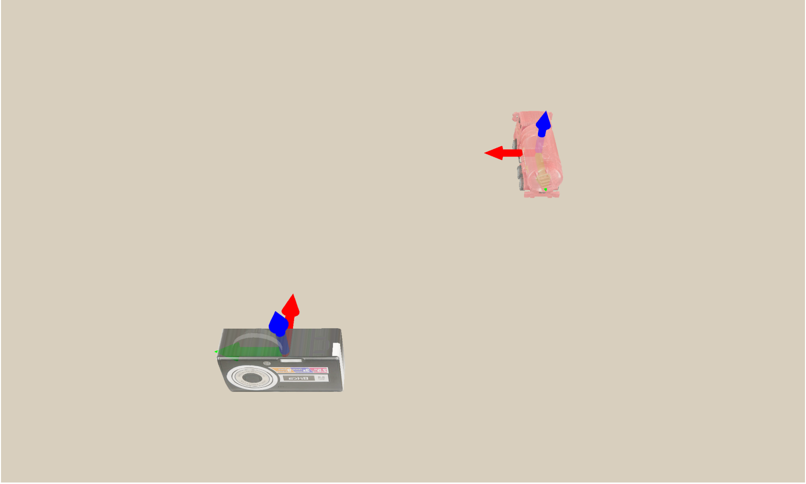
- The movement from $\mathscr{F}_1$ to $\mathscr{F}_2$ can also be represented as a linear transformation from $\mathscr{F}_1$ to $\mathscr{F}_2$, recorded by frame c, denoted as $T^c_{1\to 2}$

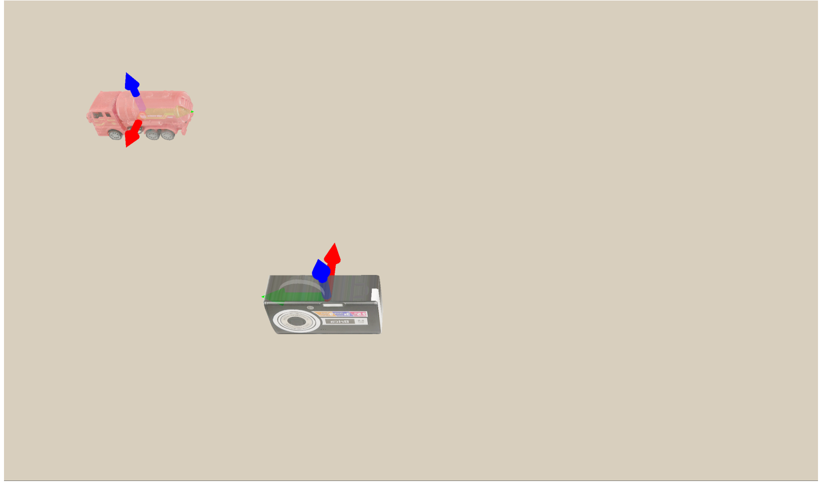
Example

- With similarity transformation:

$$T^c_{1\to 2} = T^c_{c\to 1} T^1_{1\to 2} (T^c_{c\to 1})^{-1} = \begin{bmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} & 0 & 0 \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Note: translation in $T^c_{1\to 2}$ is all zero! Why?

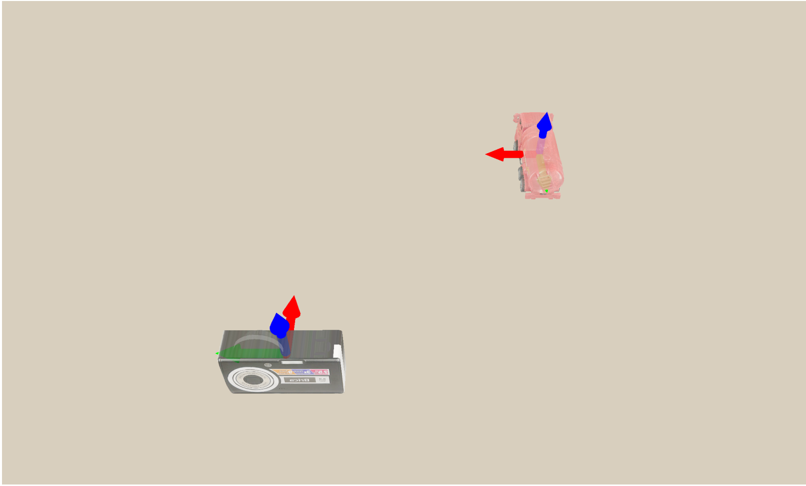


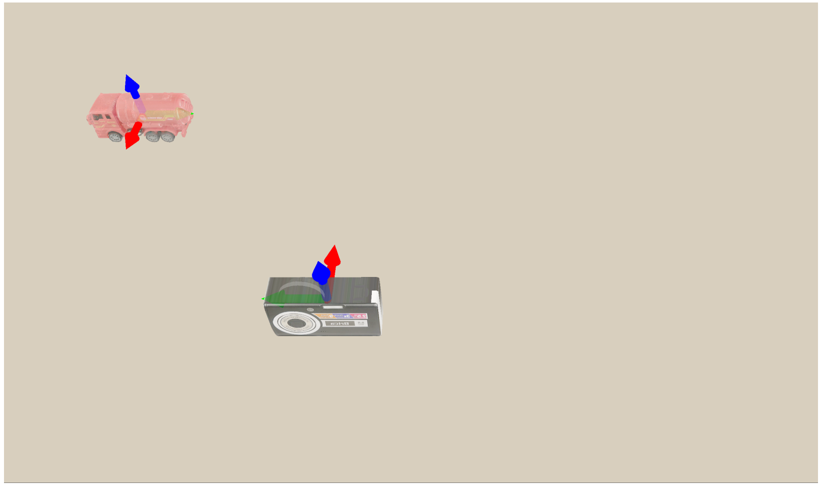


Example

- Transformation $T^c_{1\to 2}$ can be regarded as rotating about z-axis by 90 degree

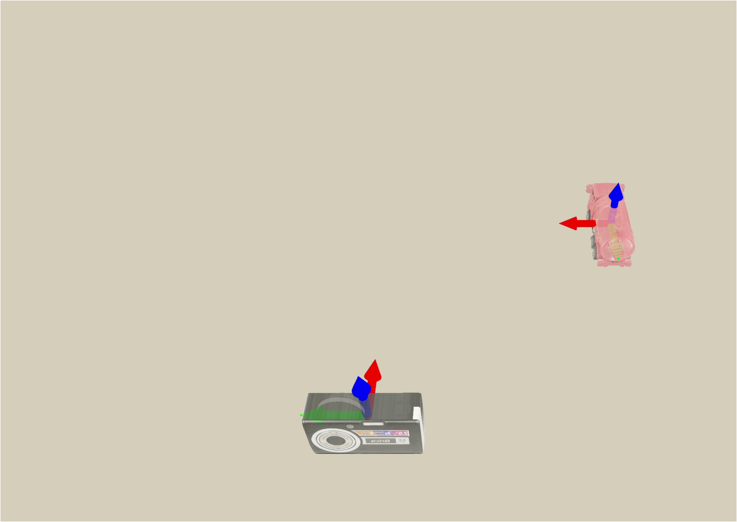
$$T^c_{1\to 2} = \begin{bmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} & 0 & 0 \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$





Example

- When observer is recording in the camera frame $\mathscr{F}_c$, the red car is rotated about the z-axis of camera frame c through +90 degree



Additional Notes by the Example

- $T^s_{1\to 2}$ is **NOT** to record the transformation by first translating $\mathscr{F}_1$ to $\mathscr{F}_2$ and then rotating (this recording convention **only** works when $\mathscr{F}_1 = \mathscr{F}_s$). It is based on the rule $T^s_{1\to 2} := T^s_{s\to 2} T^1_{1\to s}$
- An observer chooses its way to decompose $T_{1\to 2}$ into $R_{1\to 2}$ and $\mathbf{t}_{1\to 2}$ based upon its own frame choice
- We will discuss the “canonical” decomposition next time

Additional Notes by the Example

- The linear transformation view allows us to discuss the movement of bodies conveniently (without worrying about the change of observer):

$$T^s_{1\to 2} = T^s_{3\to 2} T^s_{1\to 3}$$

- Suppose a body is moving. Then,

$$T^s_{t_0\to t+\Delta t} = T^s_{t\to t+\Delta t} T^s_{t_0\to t}$$

where t parameterizes time.

Summary

- Basic notation:
  - $T^s_{s\to b}$: Record the motion of frame alignment from $\mathscr{F}_s$ to $\mathscr{F}_b$ in $\mathscr{F}_s$
- Coordinate transformation
  - $T^c_{c\to a} = T^c_{c\to b} T^b_{b\to a}$: Composition for coordinate transformation
  - $T^b_{b\to s} = (T^s_{s\to b})^{-1}$: Change of frame for $\mathscr{F}_s$ to $\mathscr{F}_b$ motion
- Linear transformation
  - $T^s_{1\to 2} := T^s_{s\to 2} T^1_{1\to s}$: Record the motion of frame alignment from $\mathscr{F}_1$ to $\mathscr{F}_2$ in $\mathscr{F}_s$
  - $T^s_{c\to a} = T^s_{b\to a} T^s_{c\to b}$: Composition as a linear transformation
- $T^s_{1\to 2} = T_{s\to b} T^b_{1\to 2} (T_{s\to b})^{-1}$: Change of frame for $\mathscr{F}_1$-to-$\mathscr{F}_2$ motion