L9: Lagrangian Dynamics

Hao Su

Spring, 2021

The flow and some contents are based on ECE5463 taught at Ohio State University by Prof. Wei Zhang

Agenda

- Lagrangian Method
- Example: Inverted Pendulum
- Example: Cart Pole
- Example: Single-Object Dynamics
- Example: Robot Arm

click to jump to the section.

Dynamics Example: Grasp

- Consider the right grasp problem

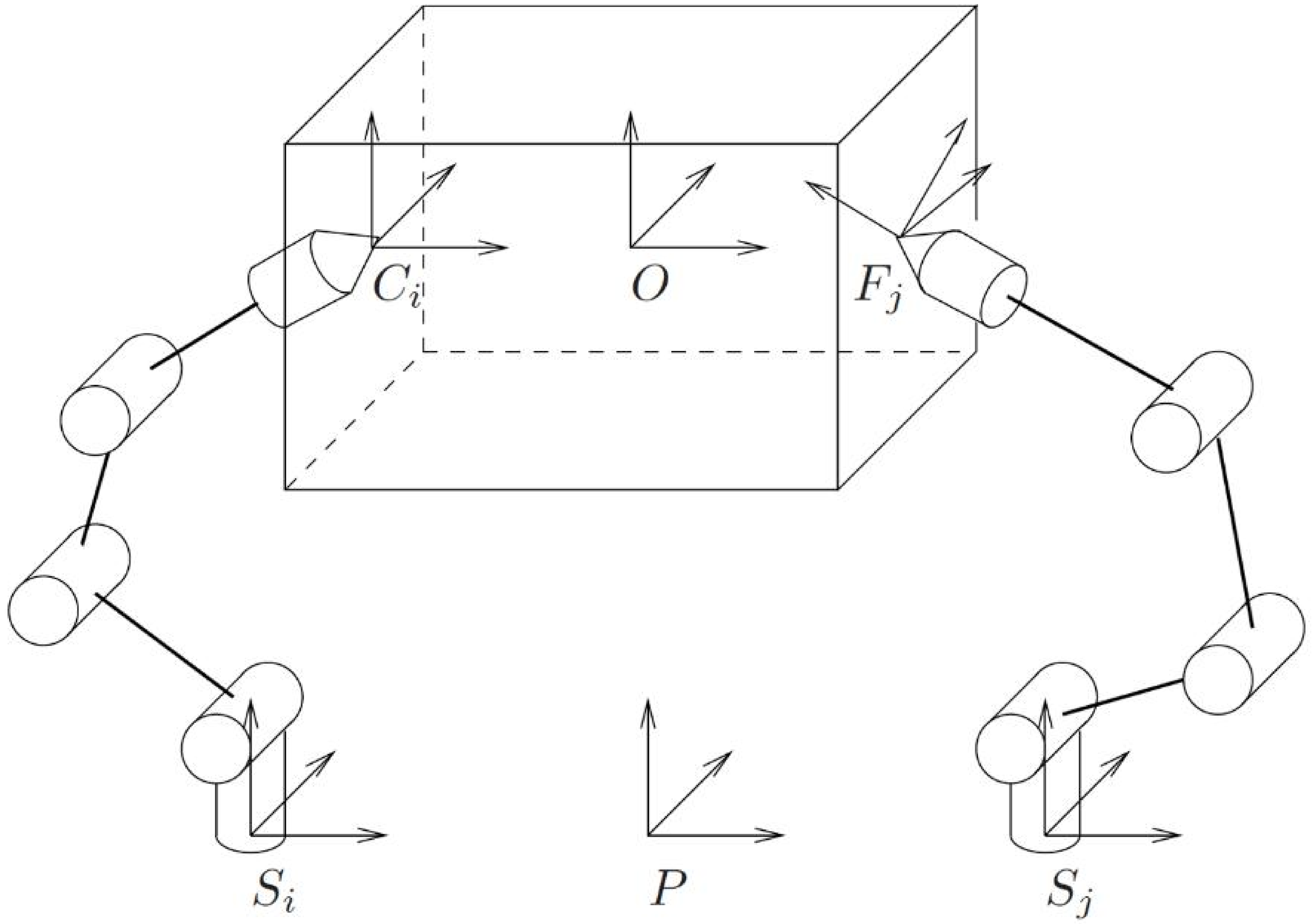
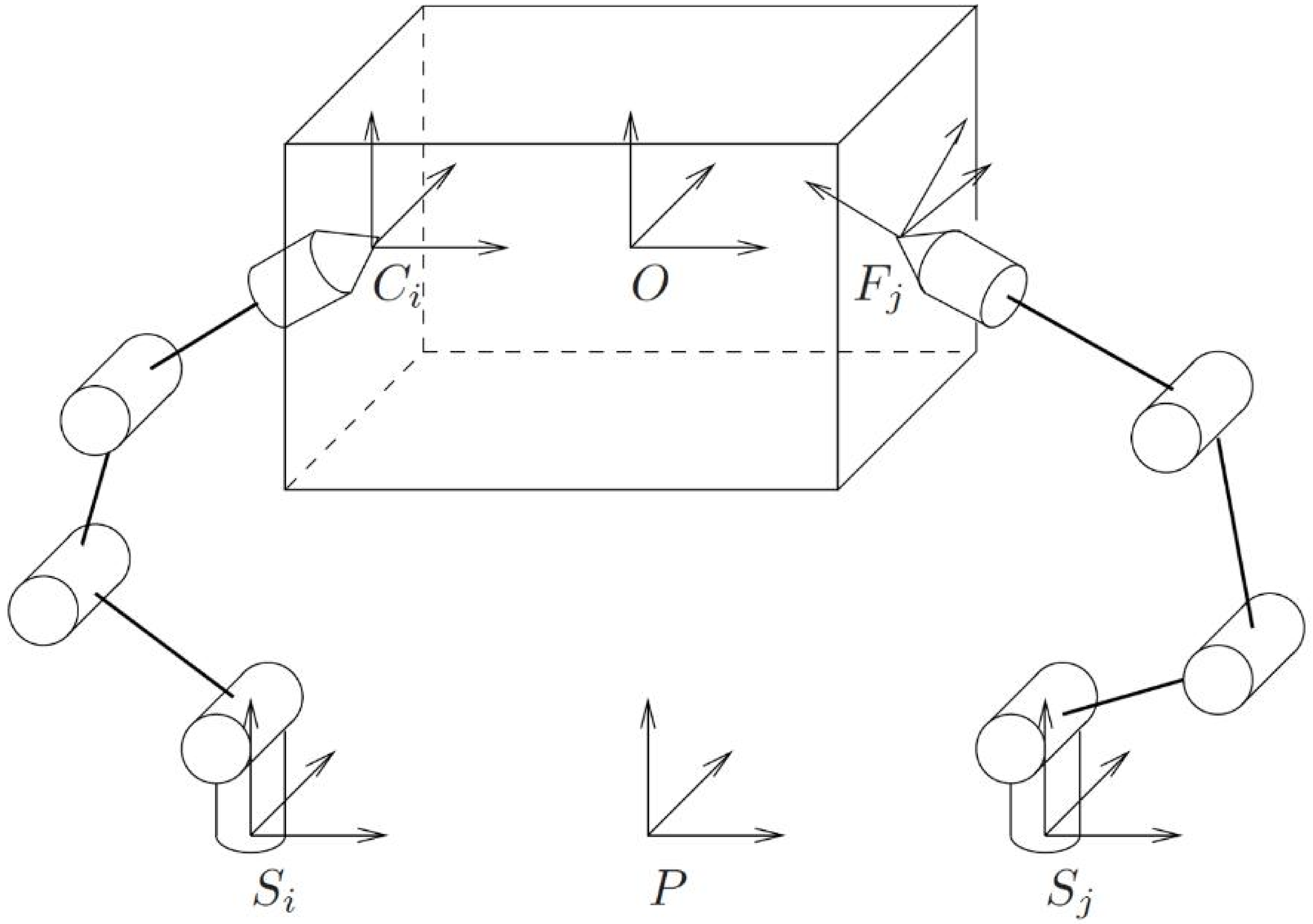


Figure 5.14: Grasp coordinate frames.

Dynamics Example: Grasp

- Consider the right grasp problem
  - Assume that we are grasping this box using two arms



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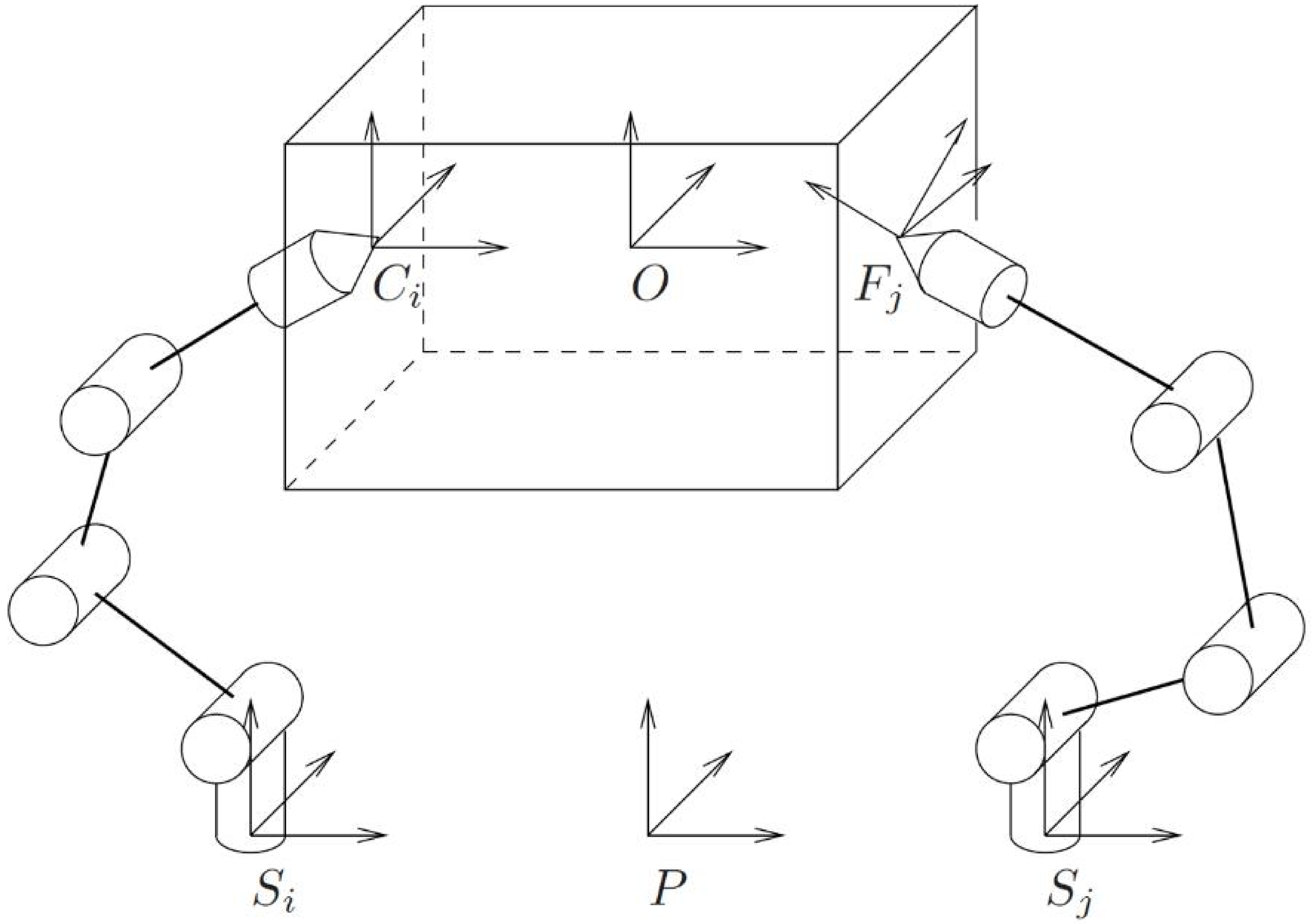

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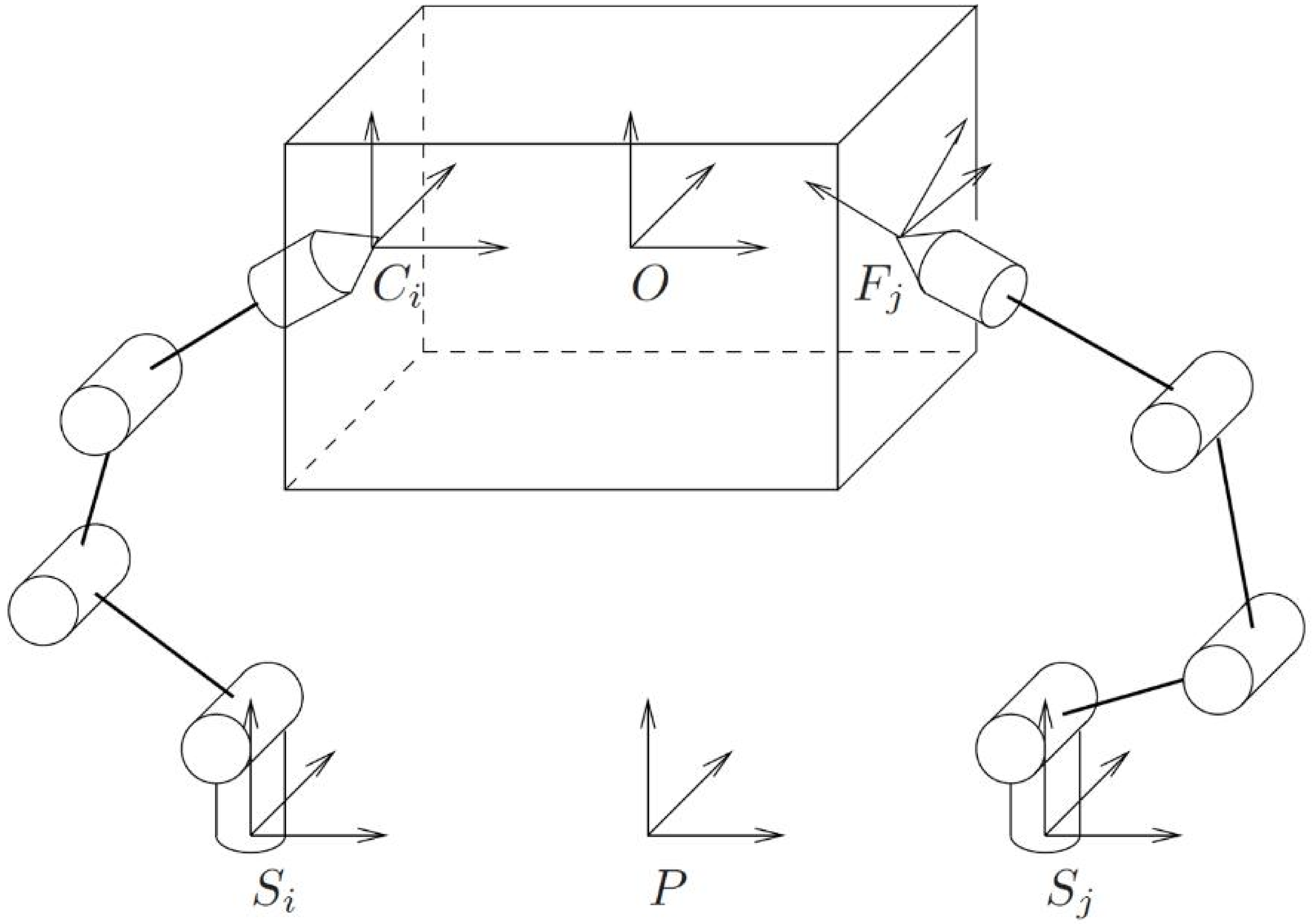

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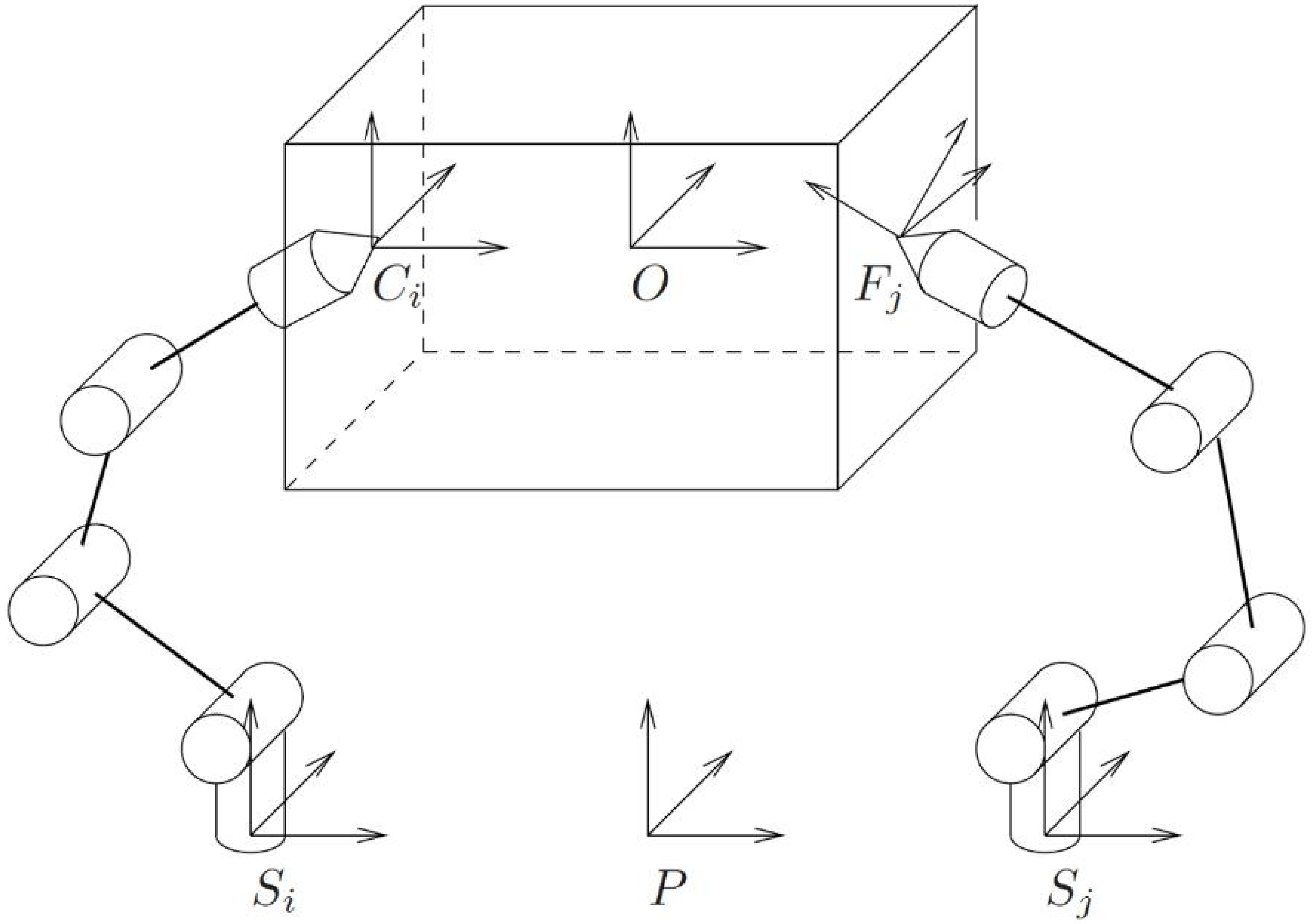

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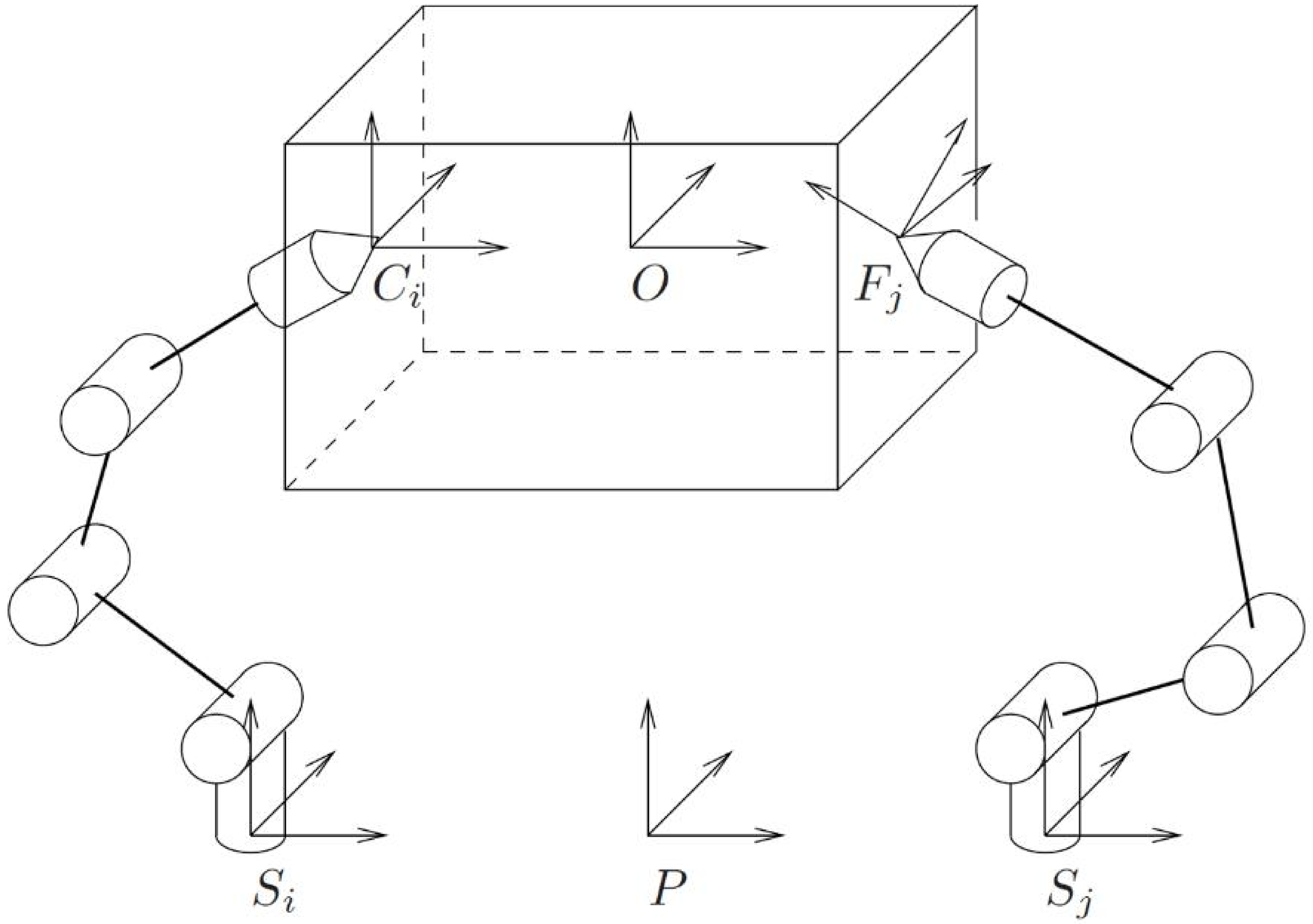
Q1: How to compute force at the tips from the torques at joints?



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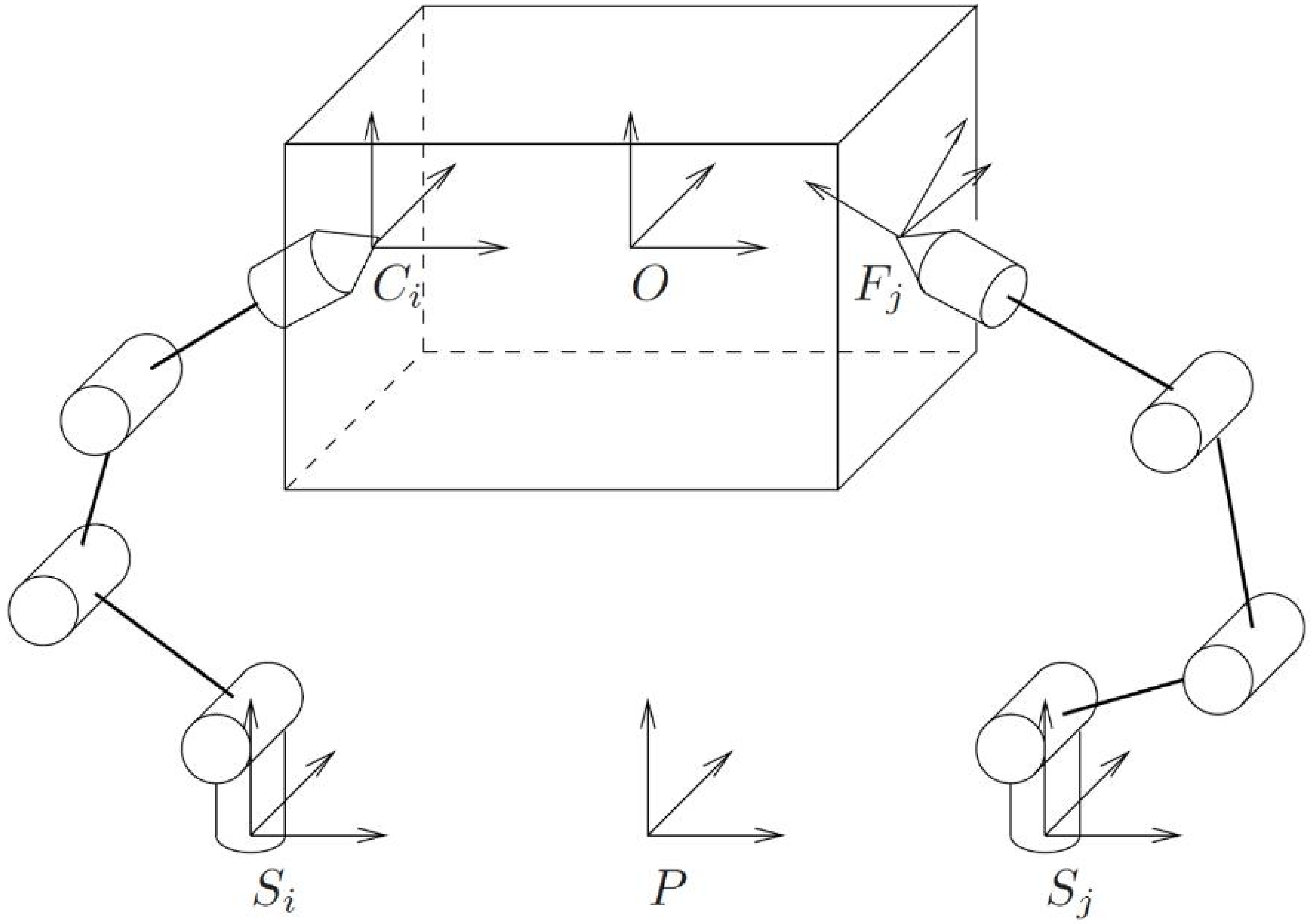
Q1: How to compute force at the tips from the torques at joints?
Q2: To keep the box static, what is the balance condition?



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Dynamics Example: Grasp

- Parameterization
    - $\theta \in \mathbb{R}^n$: vector of joint variables
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- Task
  - **Forward dynamics:** Determine acceleration $\ddot{\theta}$ given the state $(\theta, \dot{\theta})$ and the joint forces/torques

$$\ddot{\theta} = \mathrm{FD}(\tau; \theta, \dot{\theta})$$

  - **Inverse dynamics:** Finding torques/forces given state $\theta, \dot{\theta}$ and desired acceleration $\ddot{\theta}$

$$\tau = \mathrm{ID}(\ddot{\theta}; \theta, \dot{\theta})$$

Lagrangian vs. Newton-Euler Methods

- There are typically two ways to derive the equation of motion for an open-chain robot: Lagrangian method and Newton-Euler method

Lagrangian Formulation

- Energy-based method
- Often used for study of dynamic properties and analysis of control methods

Newton-Euler Formulation

- Balance of forces/torques
- Often used for numerical solution of forward/inverse dynamics

Lagrangian Method

Generalized Coordinates and Forces

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- Now consider the case in which some particles are rigidly connected, imposing constraints on their positions

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- k particles in $\mathbb{R}^3$ under n_c constraints $\Rightarrow 3k - n_c$ degree of freedom
- We introduce $n := 3k - n_c$ independent variables q_i's, called the **generalized coordinates**

$$\begin{cases} \alpha_j(p_1, \ldots, p_k) = 0 \\ j = 1, \ldots, n_c \end{cases} \quad \Leftrightarrow \quad \begin{cases} p_i = \gamma_i(q_1, \ldots, q_n) \\ i = 1, \ldots, k \end{cases}$$

Generalized Coordinates and Forces

- To describe equation of motion in terms of generalized coordinates, we also need to express external forces applied to the system in terms components along generalized coordinates. These "forces" are called *generalized forces*.

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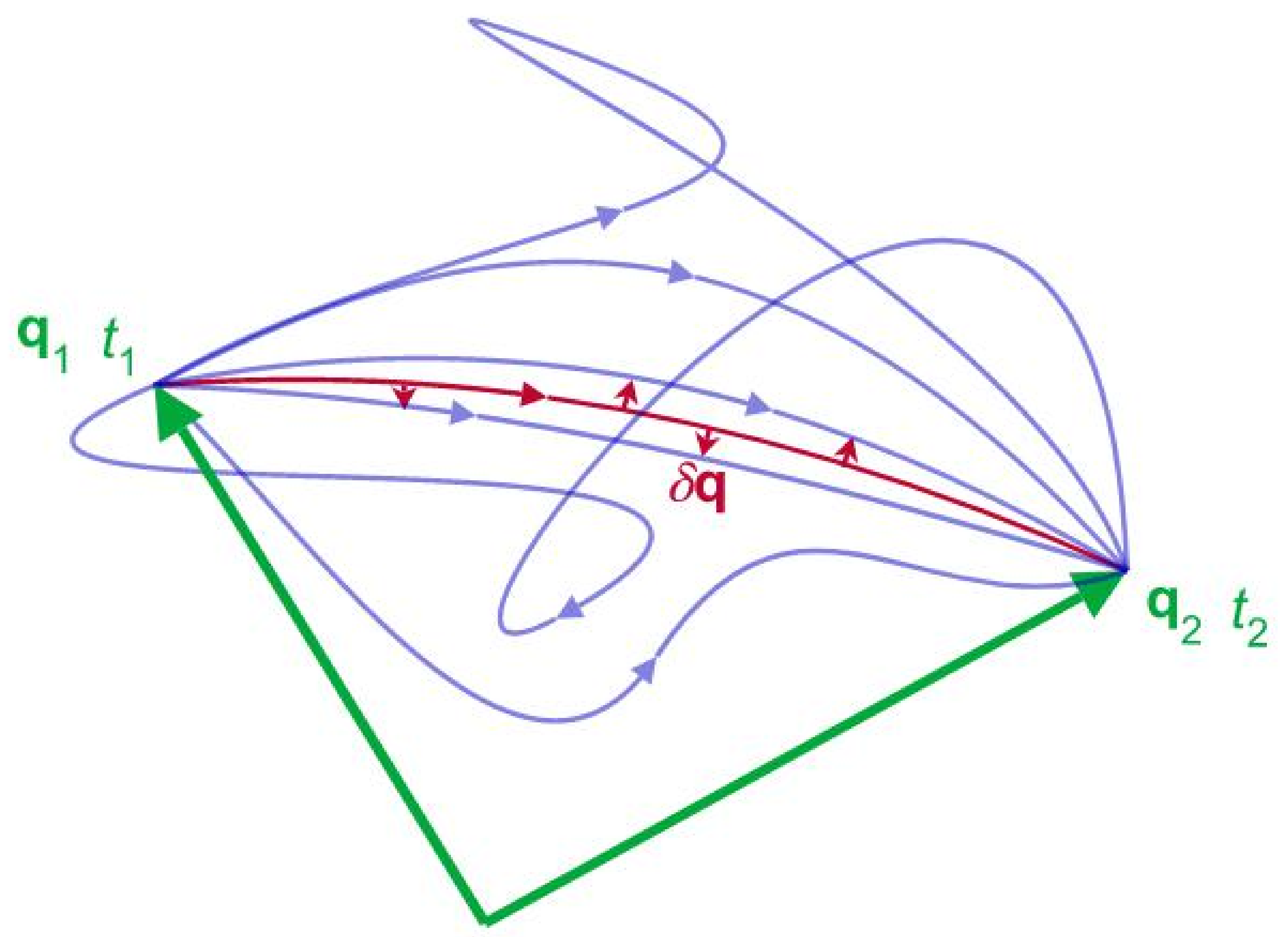
- The equation of motion of the k-particle system can thus be described in terms of $3k - n_c$ independent variables instead of the $3k$ position variables subject to n_c constraints.
- This idea of handling constraints can be extended to interconnected rigid bodies (robot arm).

Lagrangian Function

- Now let $q \in \mathbb{R}^n$ be the generalized coordinates.
- **Lagrangian function:** $L(q, \dot{q}) = T(q, \dot{q}) - V(q)$
  - $T(q, \dot{q})$: kinetic energy of system
  - $V(q)$: potential energy (given by some conservative force, e.g., gravity, electrostatic force)

The Principle of Stationary Action

- Given a pair of time instants, t_1 and t_2
- What is the curve $\boldsymbol{q} : [t_1, t_2] \to \mathcal{C}$ in the generalized coordinate space $\mathcal{C}$?



Wikipedia: Stationary Action Principle

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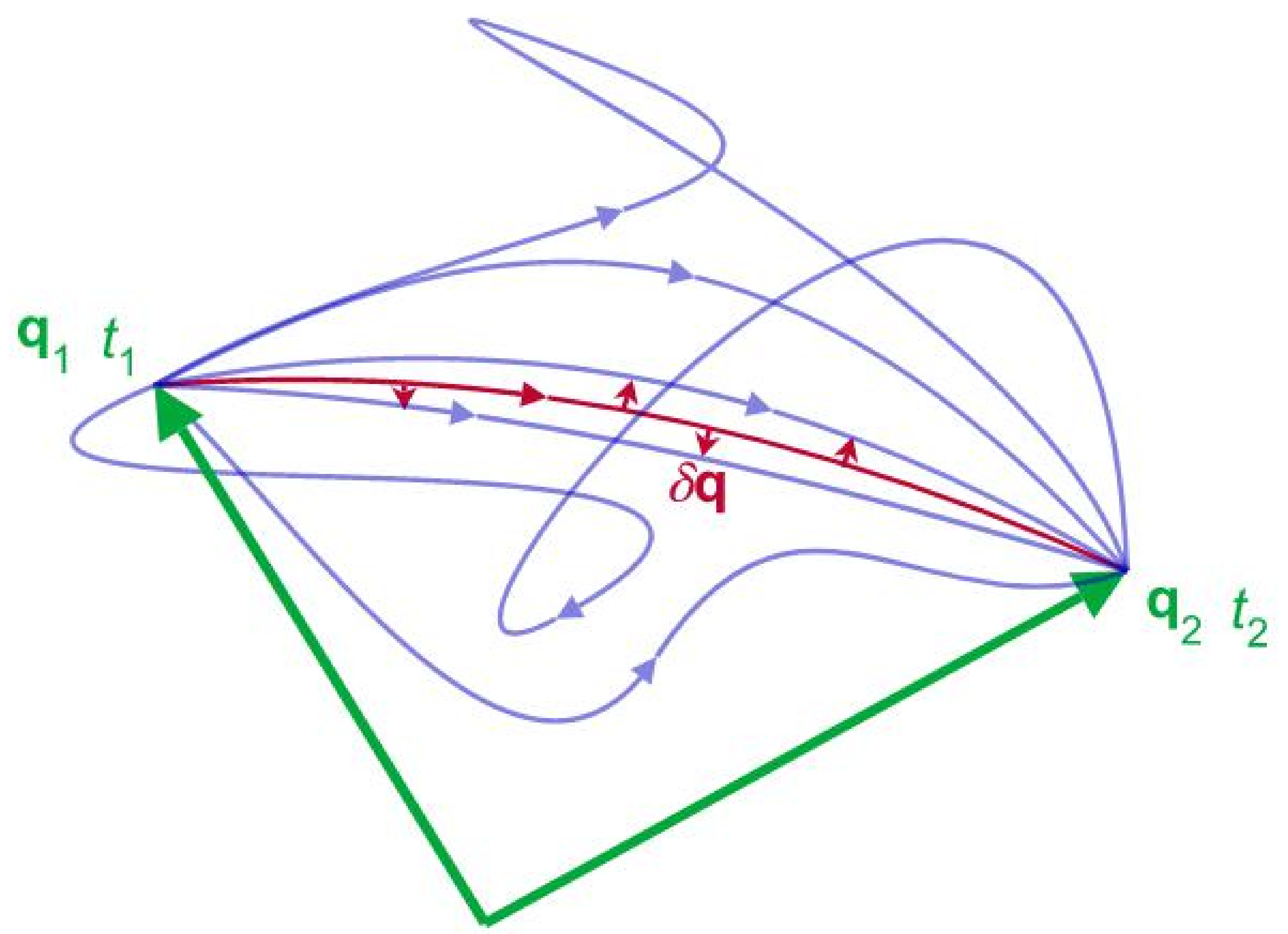
- Given a pair of time instants, t_1 and t_2
- What is the curve $\boldsymbol{q} : [t_1, t_2] \to \mathcal{C}$ in the generalized coordinate space $\mathcal{C}$?
- **Action** is defined to be a functional of $\boldsymbol{q}(t)$:

$$S[\boldsymbol{q}] = \int_{t_1}^{t_2} L(q, \dot{q})\mathrm{d}t = \int_{t_1}^{t_2} [T(\boldsymbol{q}, \dot{\boldsymbol{q}}) - V(\boldsymbol{q})]\mathrm{d}t$$



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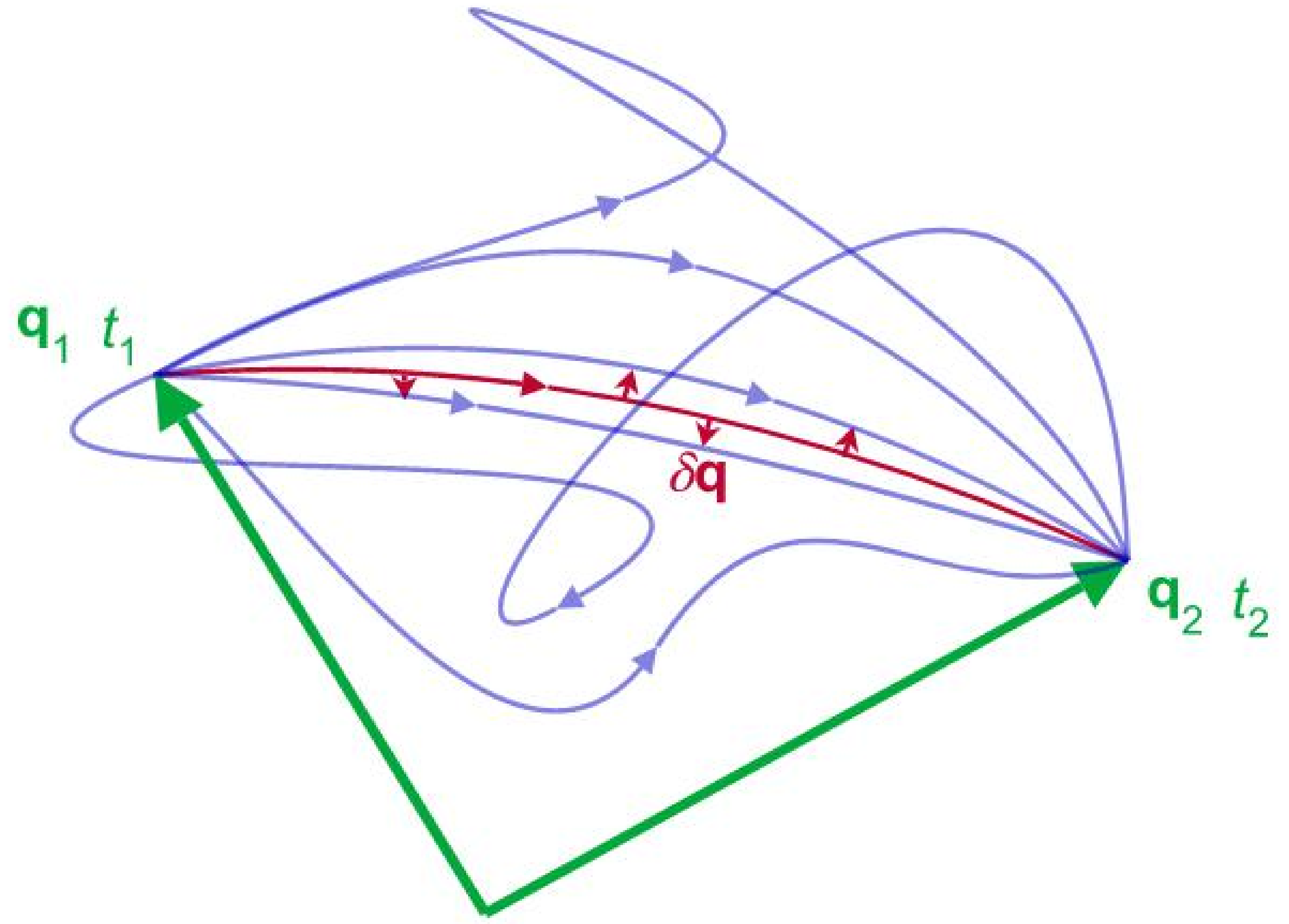
The Principle of Stationary Action

- The actual curve $\boldsymbol{q}(t)$ is a stationary point of the $S[\boldsymbol{q}]$:

$$\forall \boldsymbol{\delta} : [t_1, t_2] \to \mathcal{C}, \quad \lim_{\epsilon \to 0} \frac{1}{\epsilon}(\mathcal{S}[\boldsymbol{q} + \epsilon \boldsymbol{\delta}] - \mathcal{S}[\boldsymbol{q}]) = 0 \tag{1}$$

- Note: Treating $\boldsymbol{q}$ as a variable, and (1) is an extension of the first-order optimality condition that we use in calculus:

$$\nabla_q S[q] = 0$$



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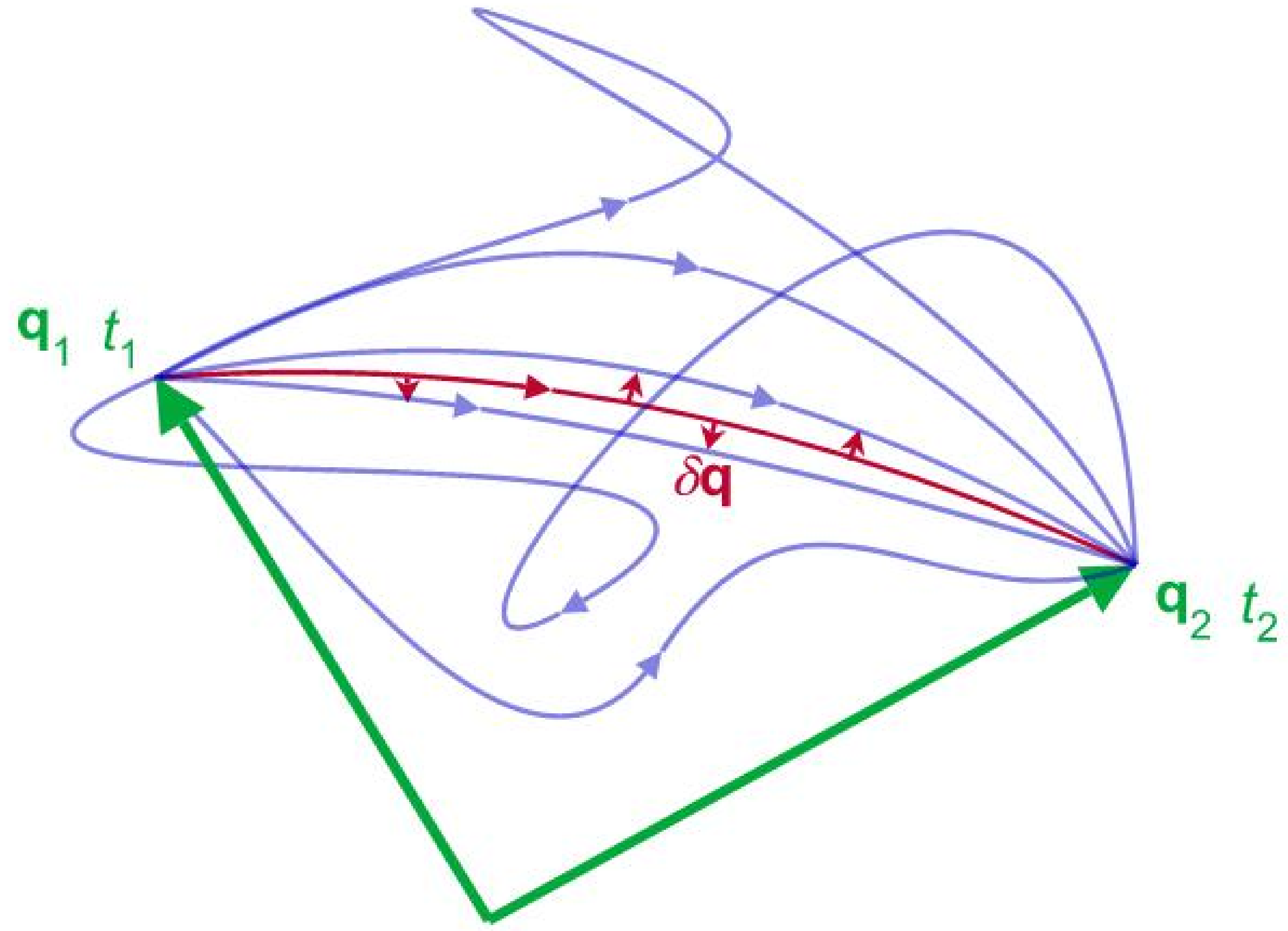
- Using *variational method*, condition (1) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0$$



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A Simple Example

- Consider a point with mass m and velocity v is falling down to the ground due to gravity, g is gravitational acceleration

$$L = \frac{1}{2}mv^2 - mgh$$

- The generalized coordinate is h and no external force are applied on this system, then

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = ma, \frac{\partial L}{\partial q} = mg$$

- Therefore we get

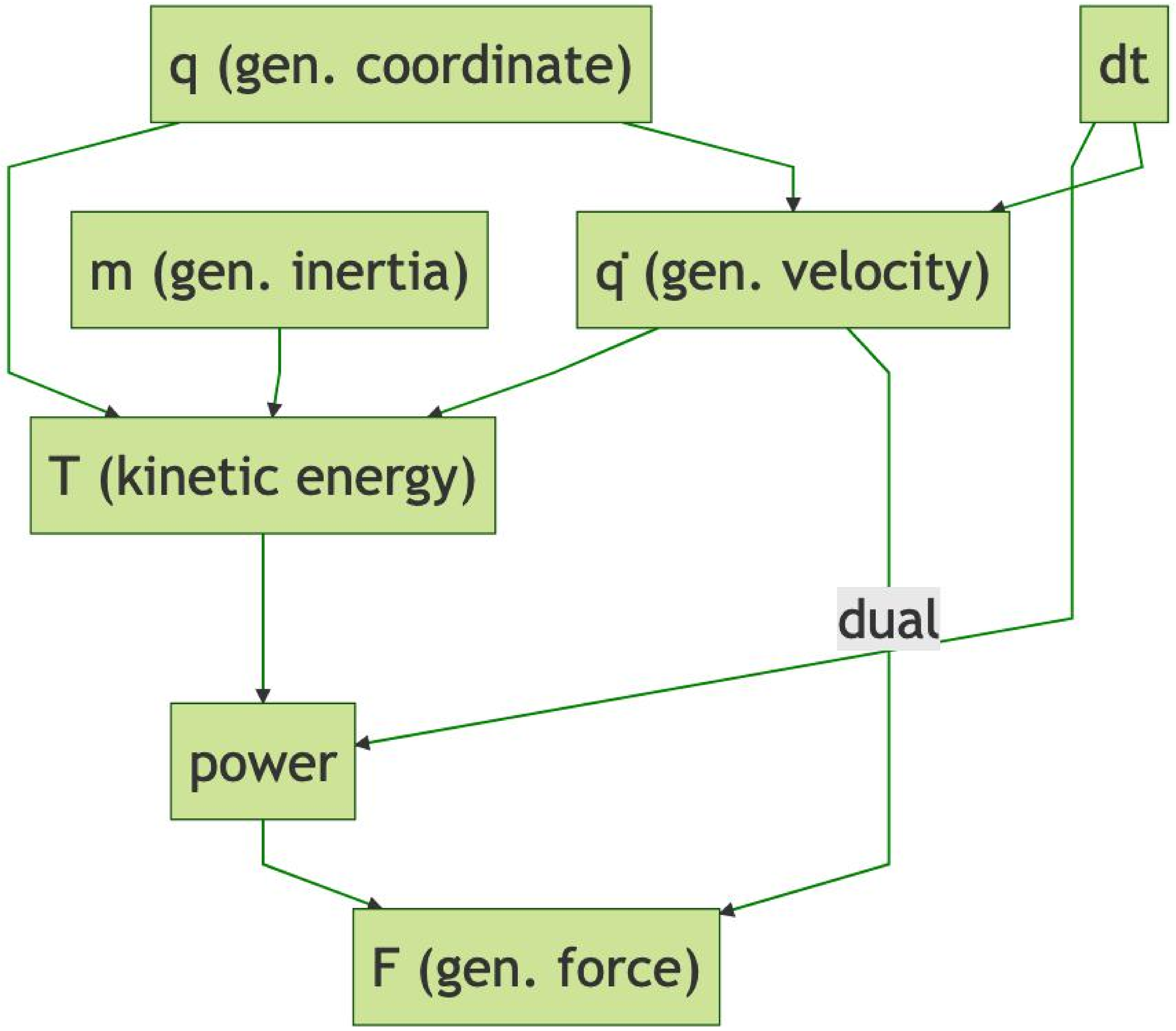
$$ma - mg = 0$$

Euler-Lagrange Equation

- When there are external non-conservative generalized force $\boldsymbol{F} \in \mathbb{R}^n$ added to the system (e.g., torque at robot arm joints), we have the following Euler-Lagrange equation:

$$\boldsymbol{F} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} \qquad \text{(Euler-Lagrange Equation)}$$

Logic behind Concepts in Lagrangian Dynamics

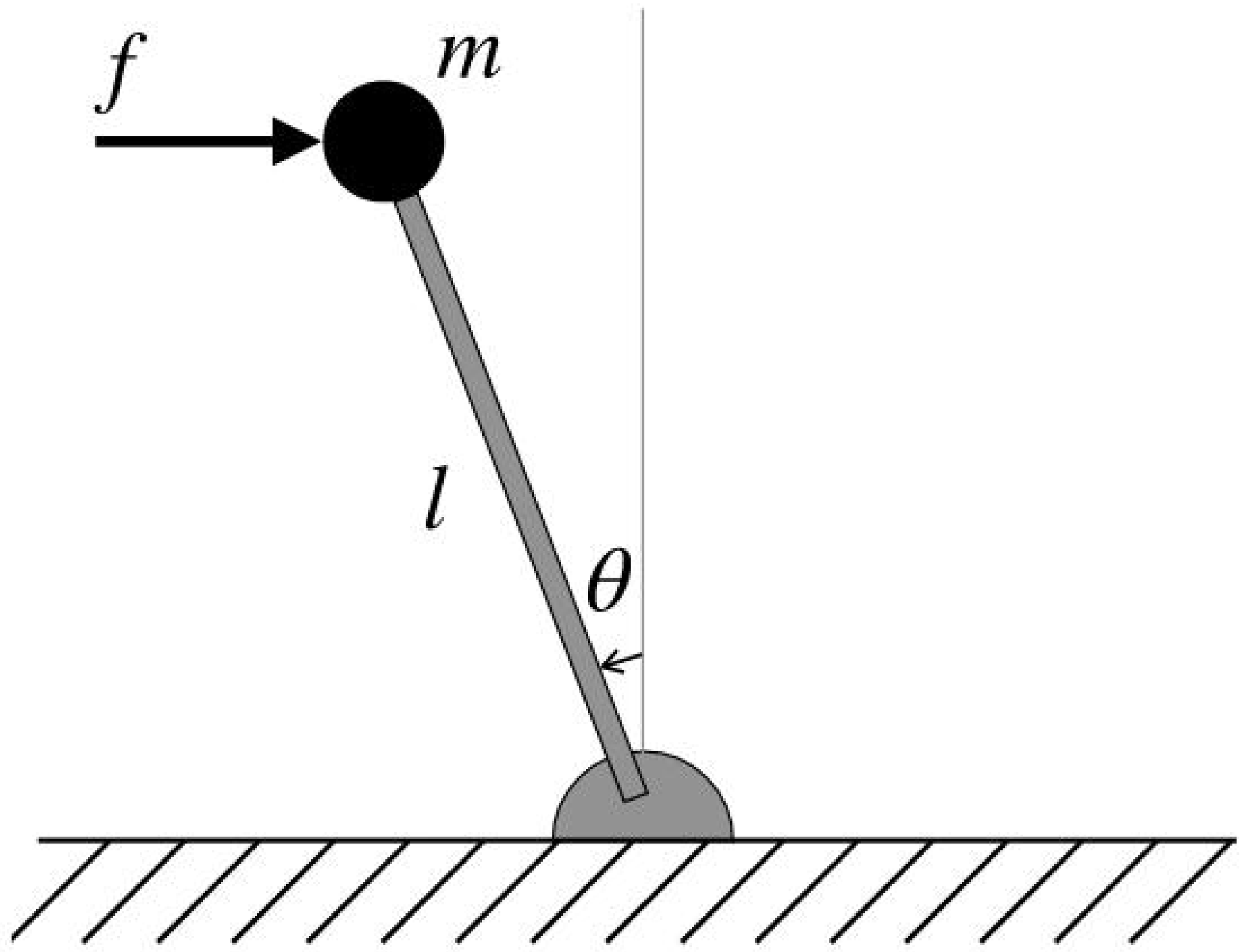


Example: Inverted Pendulum

(describe using spatial frame)

Inverted Pendulum

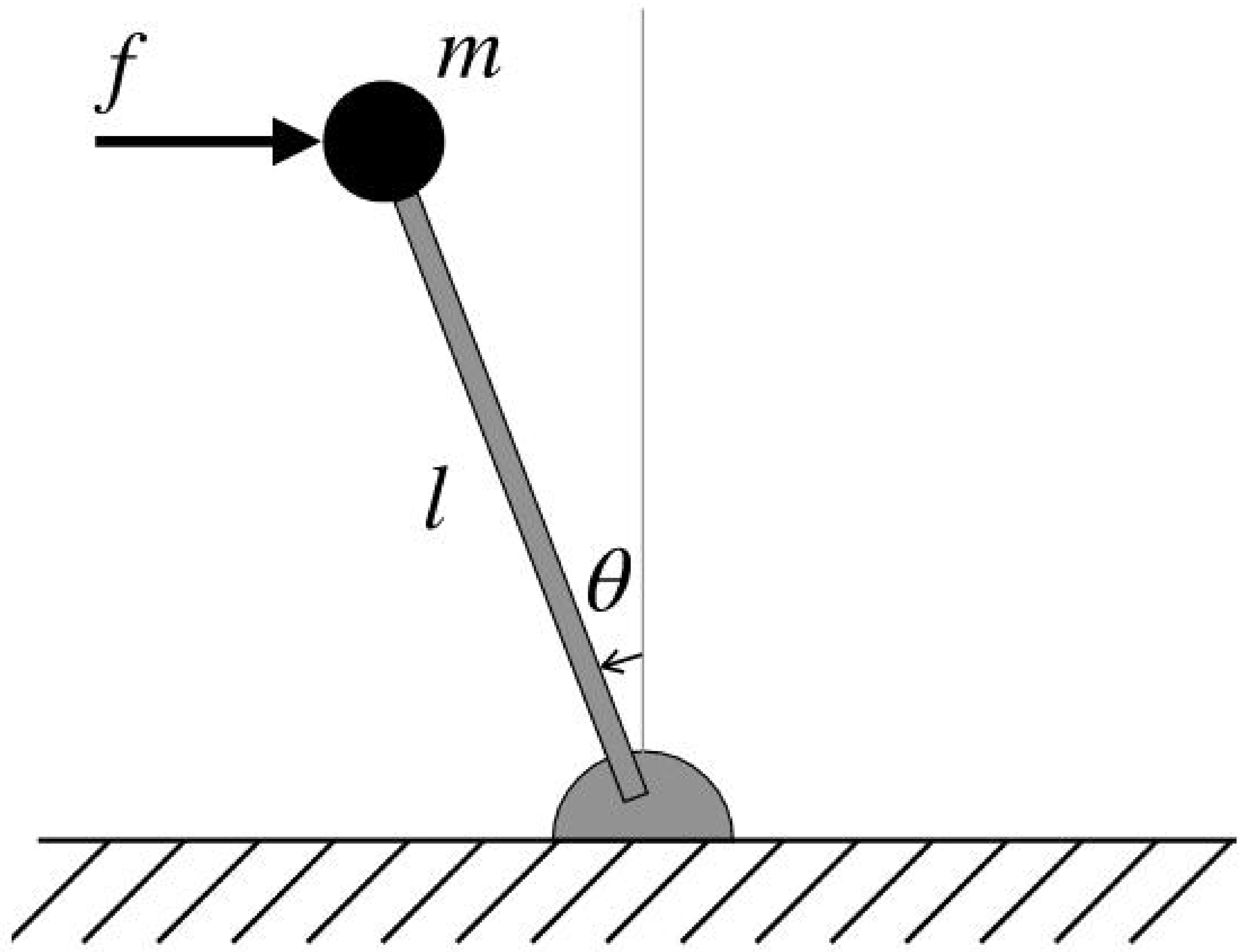
- Kinetic energy: $T = \frac{1}{2}m(\dot{\theta}l)^2$
- Potential energy: $V = mgl\cos\theta$



A schematic drawing of the inverted pendulum. The rod is considered massless.

Generalized Coordinates and Force

- The generalized coordinate of the system is θ.
- What is the generalized force?
- Recall that the inner product of generalized force and generalized velocity is the input power, so we think from the perspective of power



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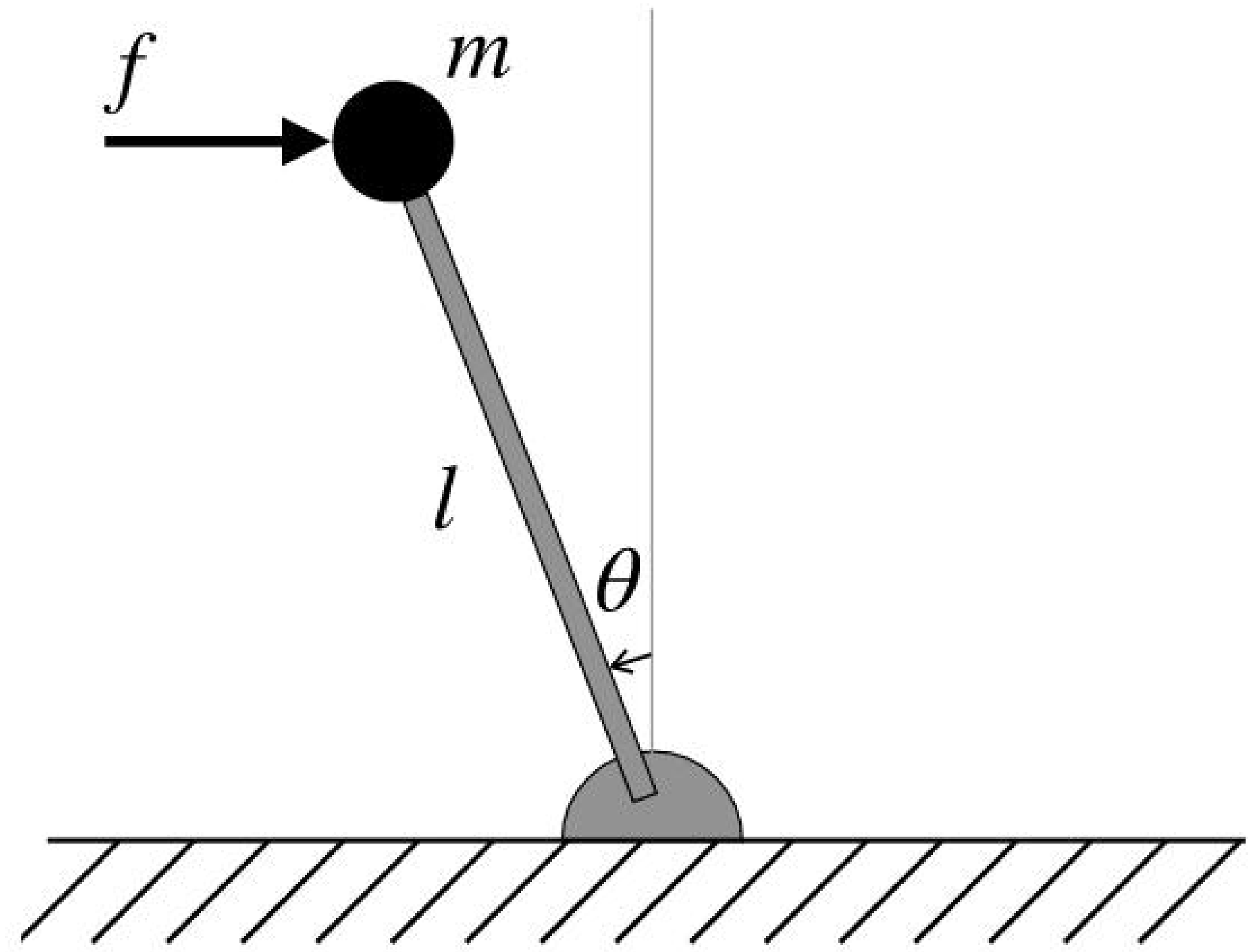
- Assume the coordinate of m is (x, y), so $P = f\frac{\mathrm{d}x}{\mathrm{d}t}$
- If F is a generalized force, then $F\dot{q} = F\frac{\mathrm{d}\theta}{\mathrm{d}t} = P = f\frac{\mathrm{d}x}{\mathrm{d}t}$
- Therefore, $F = f\frac{\mathrm{d}x}{\mathrm{d}\theta}$
- But $x = -l\sin\theta$, so $F = -fl\cos\theta$.



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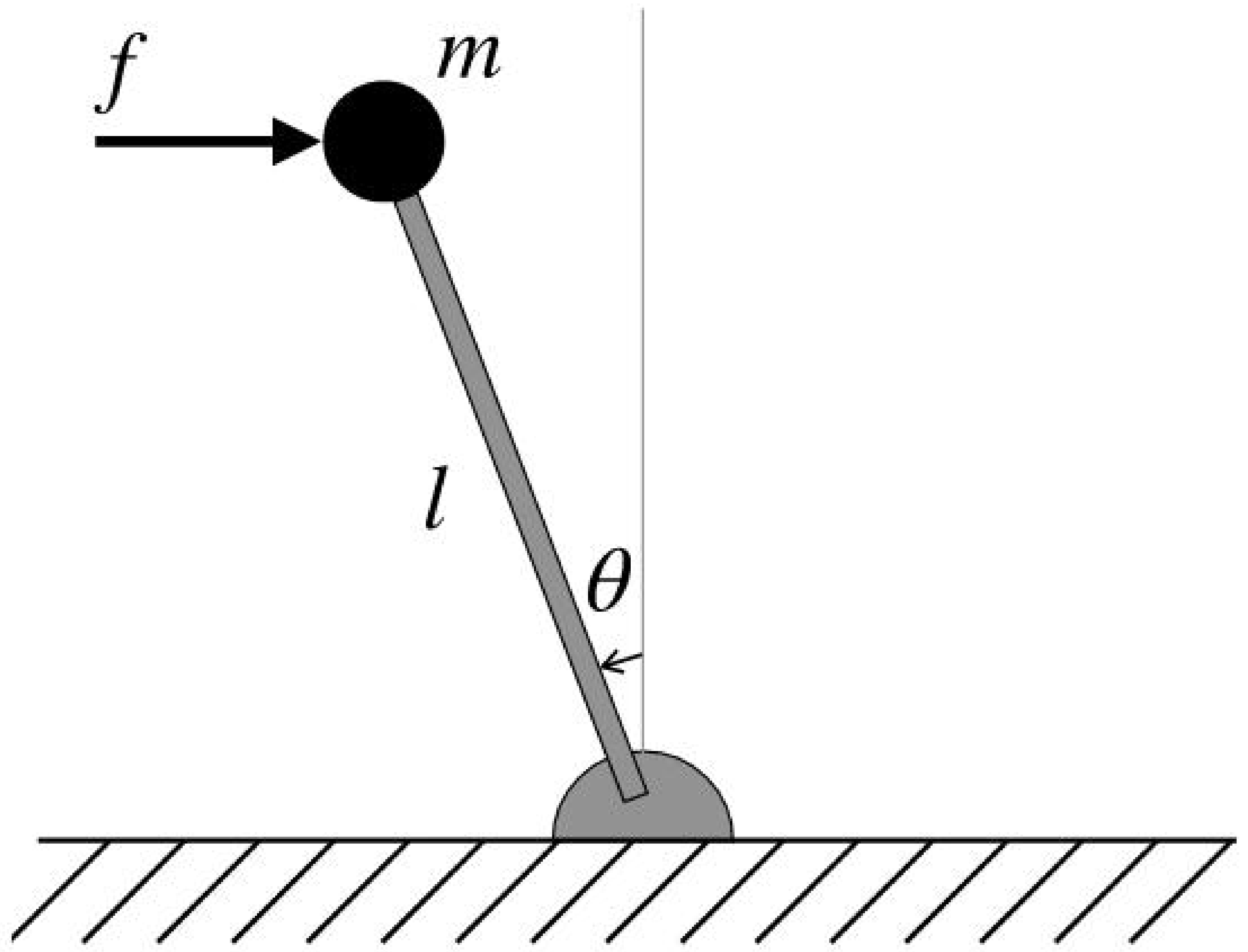
Lagrangian Equation

$$L = T - V = \frac{1}{2} m l^2 \dot{\theta}^2 - mgl\cos\theta$$
$$F = -fl\cos\theta$$

- Plug in $F = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q}$, and we have

$$ml\ddot{\theta} = -f\cos\theta + mg\sin\theta$$

- In Newton's system, the left is ma and right is the total force tangential to $\vec{l}$.



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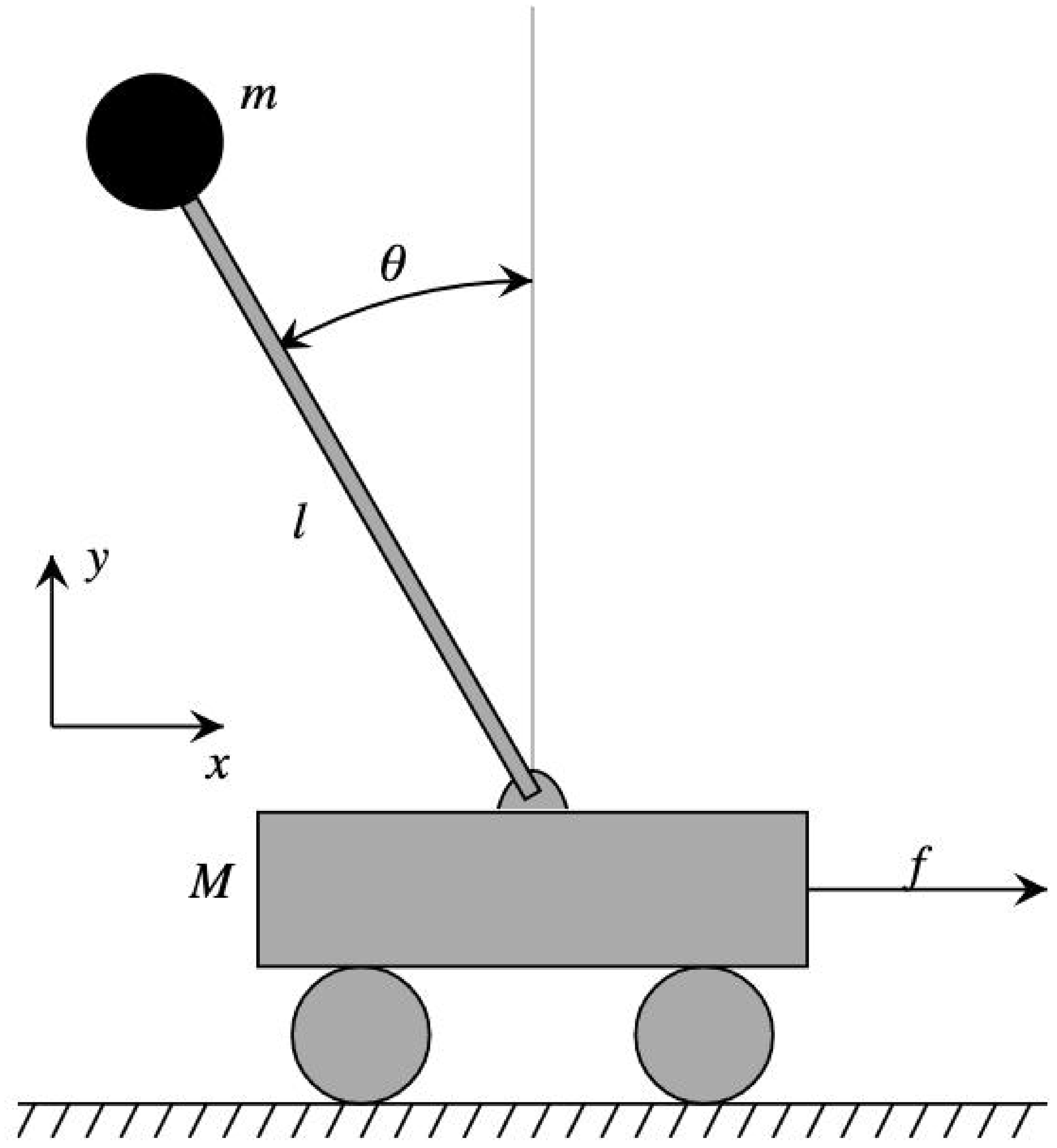
Example: Cart Pole

Cart Pole

- Kinetic energy: $T = \frac{1}{2}Mv_1^2 + \frac{1}{2}mv_2^2$
- Assume the joint position is $[x(t), 0]^T$, then
  - $v_1^2 = \dot{x}^2$
  - $v_2^2 = \left(\frac{\mathrm{d}}{\mathrm{d}t}(x - \ell \sin\theta)\right)^2 + \left(\frac{\mathrm{d}}{\mathrm{d}t}(\ell \cos\theta)\right)^2$
- Further computation shows that

$$T = \frac{1}{2}(M+m)\dot{x}^2 - m\ell\dot{x}\dot{\theta}\cos\theta + \frac{1}{2}m\ell^2\dot{\theta}^2$$

- Potential energy: $V = mgl\cos\theta$

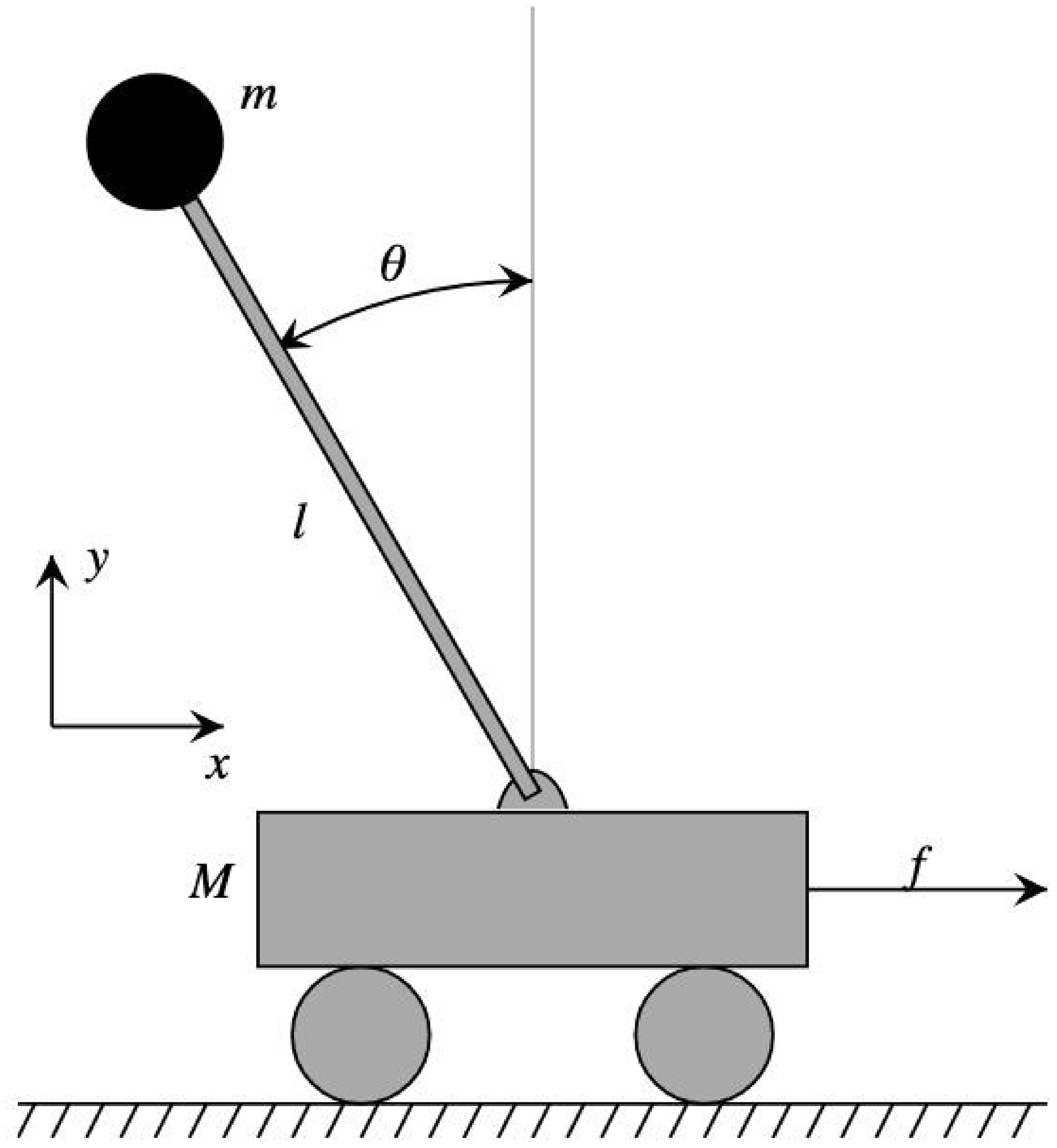


A schematic drawing of the inverted pendulum on a cart. The rod is considered massless.

https://en.wikipedia.org/wiki/Inverted_pendulum

Generalized Coordinates and Force

- First of all, note that there is an external force F, and the joint is an Underactuated joint (i.e., *no* torque at the joint)
- The generalized coordinates of the system are $q = [x, \theta]^T$, each should have a generalized force.
- What are the generalized forces?



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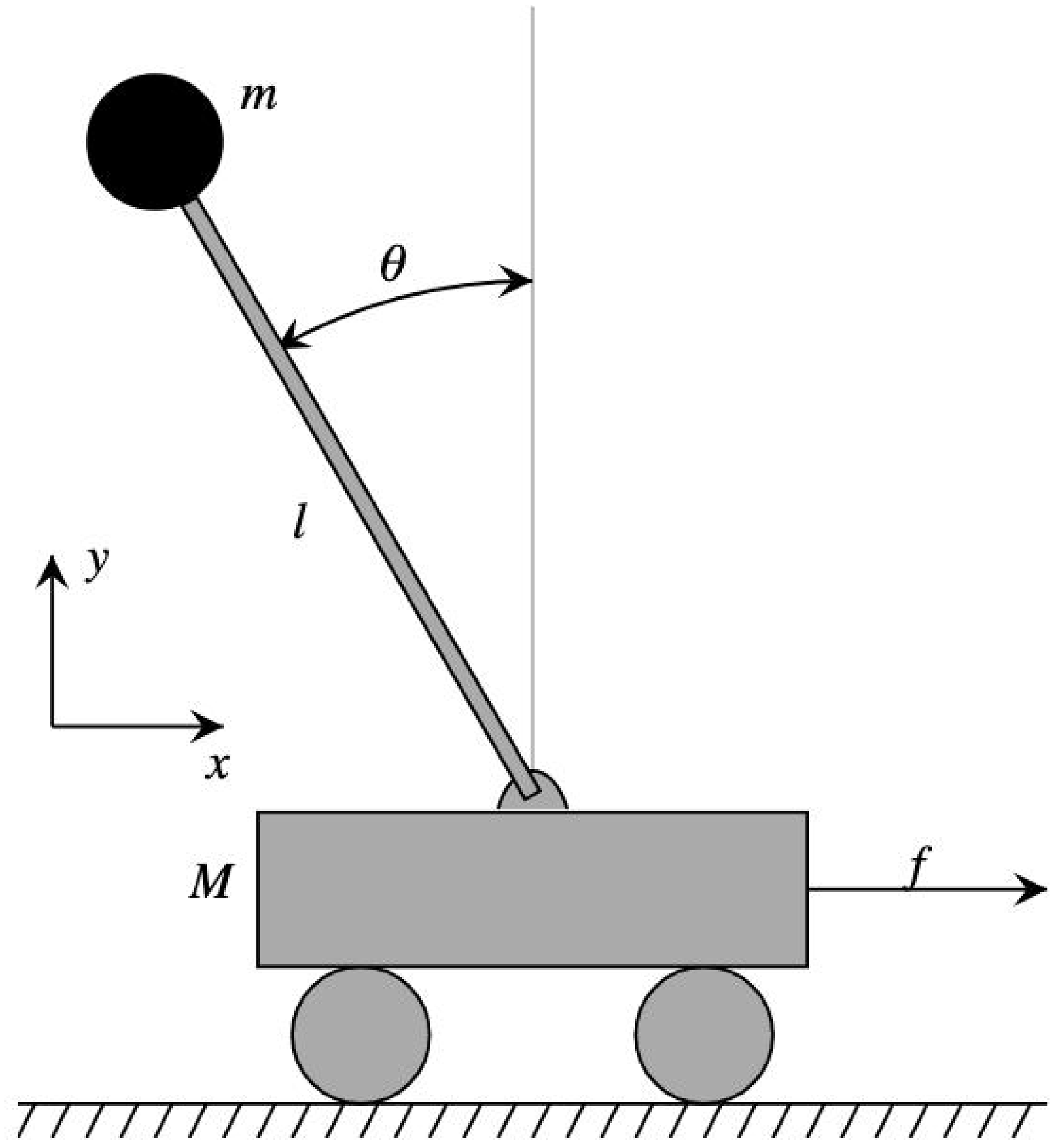
- The input power is $P = f\frac{\mathrm{d}x}{\mathrm{d}t}$
- Therefore, $[f, 0]^T$ is the generalized force dual to $[x, \theta]^T$



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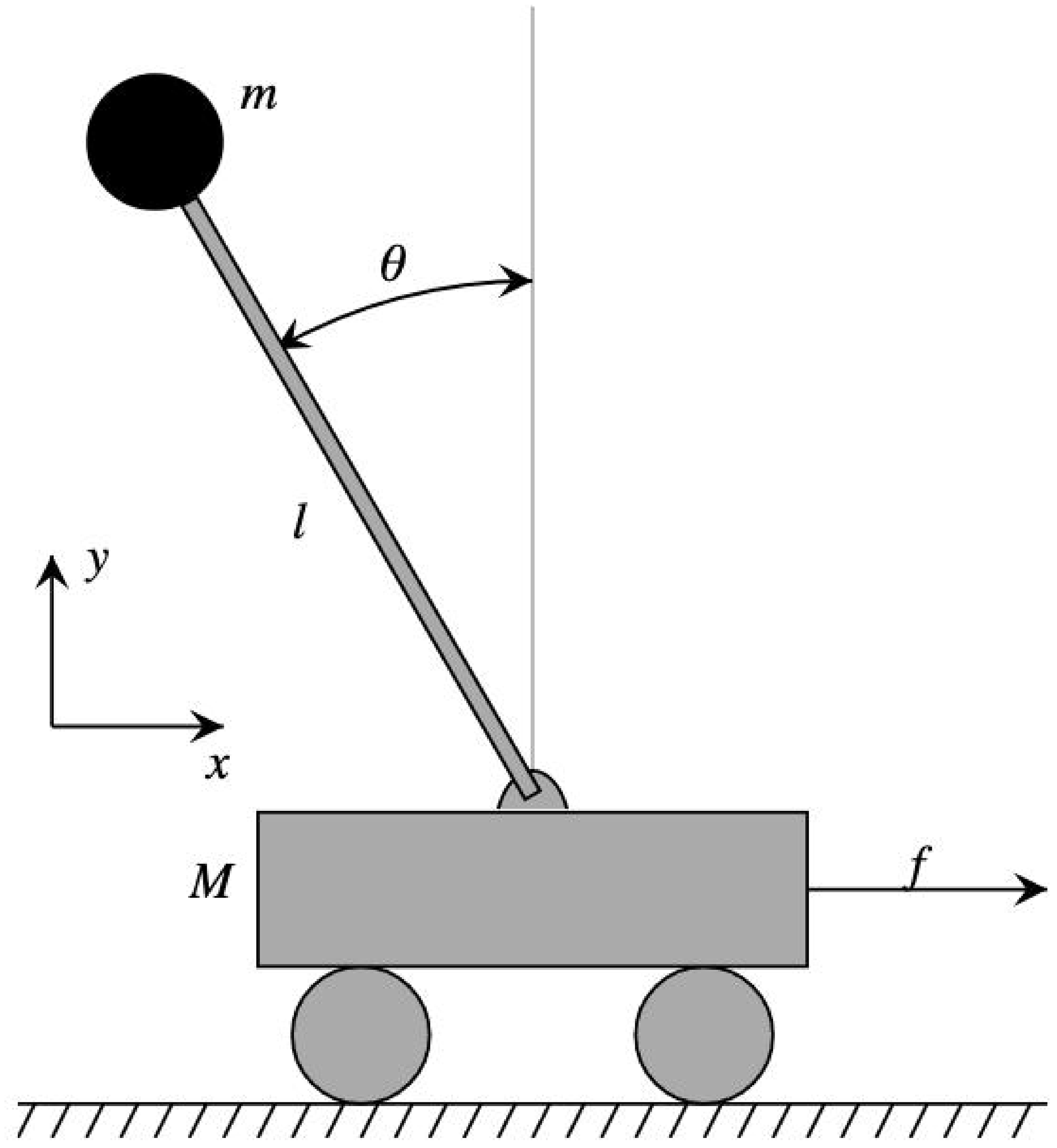
Lagrangian Equation

$$
\begin{aligned}
L &= T - V \\
&= \frac{1}{2}(M+m)\dot{x}^2 - m\ell\dot{x}\dot{\theta}\cos\theta + \frac{1}{2}m\ell^2\dot{\theta}^2 - mgl\cos\theta \\
F &= [f, 0]^T
\end{aligned}
$$

- Plug in $F = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q}$, and we have

$$
\begin{gathered}
(M+m)\ddot{x} - ml\cos\theta\ddot{\theta} + ml\sin\theta\dot{\theta}^2 = f \\
l\ddot{\theta} - g\sin\theta - \ddot{x}\cos\theta = 0
\end{gathered}
$$



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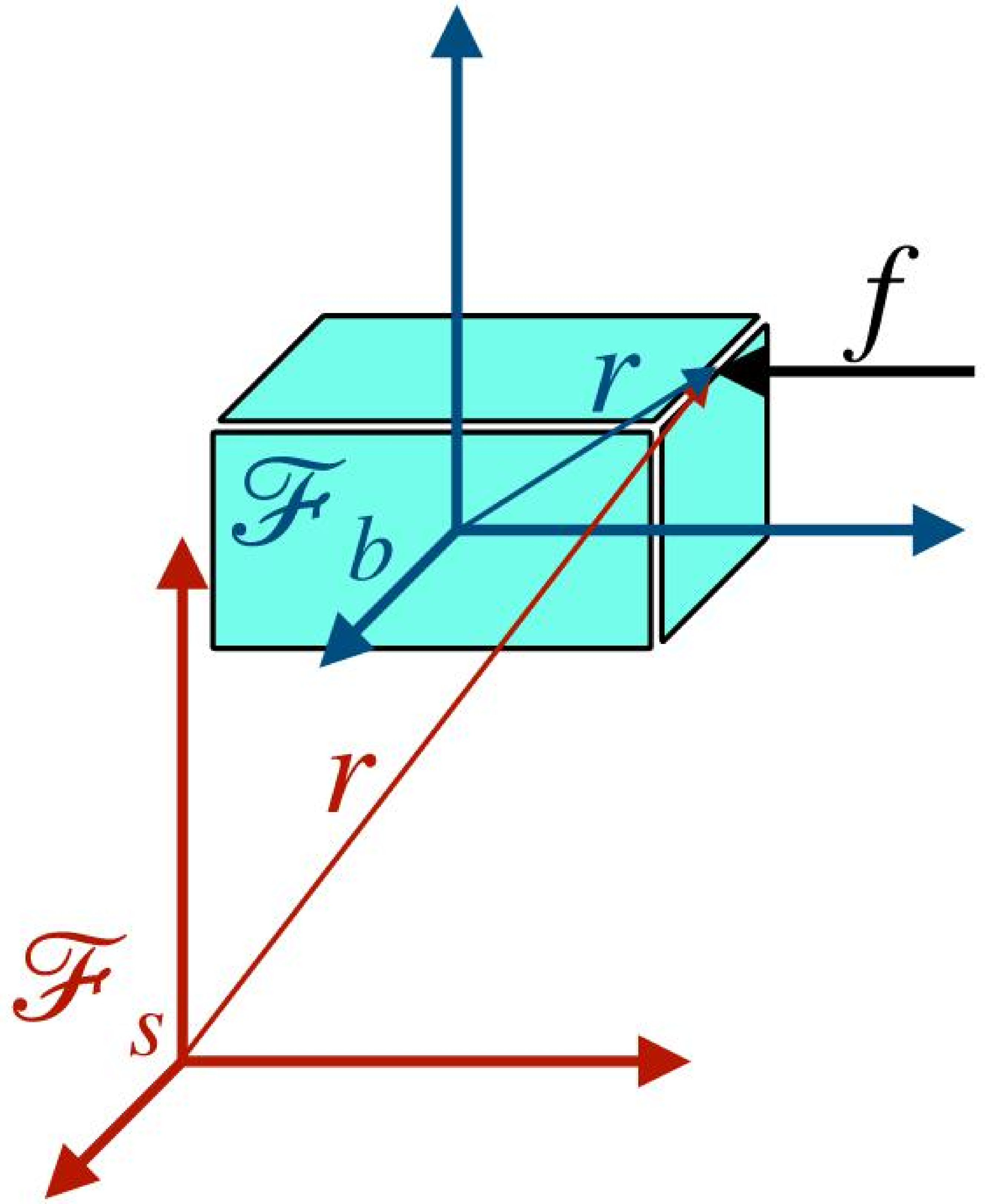
Example: Single-Object Dynamics

Setup

- Consider a moving body that is only affected by a force-torque pair (in the sense of the conventional force and torque)



Prep: Derivative of Acceleration

- Like for velocity, we use the following rule to compute the gradient of velocity in an arbitrary observer's frame:

$$\boldsymbol{a}_{b(t_0)}^{o(t_0)} = \frac{\mathrm{d}}{\mathrm{d}t} \boldsymbol{v}_{s(t_0)\to b(t)}^{o(t_0)} \Big|_{t=t_0}$$

- Different from the definition of velocity, acceleration only has one subscript
- We clone a frame $s(t_0)$ when taking the derivative, so the definition of acceleration is invariant to $s(t)$

Prep: Derivative of Acceleration

- Spatial acceleration (spatial frame is an inertia frame): $\boldsymbol{a}_{b(t)}^{s(t)} = \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{v}_{s(t_0)\to b(t)}^{s(t_0)}\Big|_{t=t_0}$
- Body acceleration:

$$\because \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{v}_{s(t_0)\to b(t)}^{b(t)} = \frac{\mathrm{d}}{\mathrm{d}t}R_{b(t)\to b(t_0)}^{b(t)}\boldsymbol{v}_{s(t_0)\to b(t)}^{b(t_0)} = -[\boldsymbol{\omega}^{b(t)}]\boldsymbol{v}_{s(t_0)\to b(t)}^{b(t_0)} + \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{v}_{s(t_0)\to b(t)}^{b(t_0)}$$

$$\therefore \boxed{\boldsymbol{a}_{b(t)}^{b(t)} = \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{v}_{s(t_0)\to b(t)}^{b(t_0)}\Big|_{t=t_0} = \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{v}_{s(t_0)\to b(t)}^{b(t)}\Big|_{t=t_0} + [\boldsymbol{\omega}^{b(t_0)}]\boldsymbol{v}_{s(t_0)\to b(t)}^{b(t_0)}}$$

- Using $\boldsymbol{v}_{s(t)\to b(t)}^{s(t)} = R_{s(t)\to b(t)}^{s(t)}\boldsymbol{v}_{s(t)\to b(t)}^{b(t)}$, you can verify that $R_{b(t)\to s(t)}^{b(t)}\boldsymbol{a}^{s(t)} = \boldsymbol{a}^{b(t)}$ (so that $\boldsymbol{f}^o = m\boldsymbol{a}^o$ for both the spatial and body frames).
- The second term in the body-frame acceleration is the *Coriolis acceleration*.

Body-Frame Lagrangian Derivation

- Recall that the kinetic energy T is:

$$T = \frac{1}{2}(\boldsymbol{\xi}^{b(t)}_{s(t)\to b(t)})^T \mathfrak{M}^b \boldsymbol{\xi}^{b(t)}_{s(t)\to b(t)}$$

where $\mathfrak{M}^b = \begin{bmatrix} m\mathrm{Id}_{3\times 3} & 0 \\ 0 & \boldsymbol{I}^b \end{bmatrix} \in \mathbb{R}^{6\times 6}$ and $\boldsymbol{\xi}^b = \begin{bmatrix} \boldsymbol{v}^b \\ \boldsymbol{\omega}^b \end{bmatrix}$

Generalized Velocity and Force

- Recall that we introduced $\boldsymbol{F}^{b(t)} = \begin{bmatrix} \boldsymbol{f}^{b(t)} \\ \boldsymbol{\tau}^{b(t)} \end{bmatrix}$ and $(\boldsymbol{F}^{b(t)})^T \boldsymbol{\xi}^{b(t)} = \frac{\mathrm{d}T}{\mathrm{d}t}$
  - implies that $(\boldsymbol{\xi}^{b(t)}, \boldsymbol{F}^{b(t)})$ is a dual pair
  - $\boldsymbol{F}^{b(t)}$ is a generalized force and $\boldsymbol{\xi}^{b(t)}$ is a generalized velocity

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  - implies that $(\boldsymbol{\xi}^{b(t)}, \boldsymbol{F}^{b(t)})$ is a dual pair
  - $\boldsymbol{F}^{b(t)}$ is a generalized force and $\boldsymbol{\xi}^{b(t)}$ is a generalized velocity
- Therefore, we can plug them in Euler-Lagrange equation:

$$F = \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} \qquad \text{(Euler-Lagrange Equation)}$$

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- Recall that we have derived that

$$L = \frac{1}{2}(\boldsymbol{\xi}^{b(t)})^T \mathfrak{M}^{b(t)} \boldsymbol{\xi}^{b(t)} = \frac{1}{2} m \|\boldsymbol{v}^{b(t)}\|^2 + \frac{1}{2}(\boldsymbol{\omega}^{b(t)})^T \boldsymbol{I}^b \boldsymbol{\omega}^{b(t)}$$

- Therefore, $\frac{\partial L}{\partial \boldsymbol{v}^{b(t)}} = m\boldsymbol{v}^{b(t)}$, $\frac{\partial L}{\partial \boldsymbol{\omega}^{b(t)}} = \boldsymbol{I}^b \boldsymbol{\omega}^{b(t)}$

Body-Frame Lagrangian Derivation

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \boldsymbol{\xi}^{b(t)}} = \boldsymbol{F}^{b(t)}, \qquad \frac{\partial L}{\partial \boldsymbol{v}^{b(t)}} = m\boldsymbol{v}^{b(t)}, \qquad \frac{\partial L}{\partial \boldsymbol{\omega}^{b(t)}} = \boldsymbol{I}^{b}\boldsymbol{\omega}^{b(t)}$$

- Therefore,

$$\boldsymbol{f}^{b(t_0)} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \boldsymbol{v}^{b(t_0)}} = m\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{v}^{b(t_0)}_{s(t_0)\to b(t)} = m\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{v}^{b(t)}_{s(t_0)\to b(t)}\bigg|_{t=t_0} + m[\boldsymbol{\omega}^{b(t_0)}_{s(t_0)\to b(t_0)}]\boldsymbol{v}^{b(t_0)}_{s(t_0)\to b(t_0)}$$

$$\boldsymbol{\tau}^{b(t_0)} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \boldsymbol{\omega}^{b(t_0)}} = \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{I}^{b}\boldsymbol{\omega}^{b(t)}_{s(t_0)\to b(t)}\bigg|_{t=t_0} = \boldsymbol{I}^{b}\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\omega}^{b(t)}_{s(t_0)\to b(t)}\bigg|_{t=t_0} + [\boldsymbol{\omega}^{b(t_0)}]\boldsymbol{I}^{b}\boldsymbol{\omega}^{b(t_0)}$$

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$$\boldsymbol{\tau}^{b(t_0)} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \boldsymbol{\omega}^{b(t_0)}} = \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{I}^b\boldsymbol{\omega}^{b(t)}_{s(t_0)\to b(t)}\bigg|_{t=t_0} = \boldsymbol{I}^b\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\omega}^{b(t)}_{s(t_0)\to b(t)}\bigg|_{t=t_0} + [\boldsymbol{\omega}^{b(t_0)}]\boldsymbol{I}^b\boldsymbol{\omega}^{b(t_0)}$$

- In matrix form, we have the famous **body-frame Newton-Euler equation**:

$$\begin{bmatrix} m\mathrm{Id}_{3\times 3} & 0 \\ 0 & \boldsymbol{I}^b \end{bmatrix}\begin{bmatrix} \dot{\boldsymbol{v}}^b \\ \dot{\boldsymbol{\omega}}^b \end{bmatrix} + \begin{bmatrix} \boldsymbol{\omega}^b \times m\boldsymbol{v}^b \\ \boldsymbol{\omega}^b \times \boldsymbol{I}^b\boldsymbol{\omega}^b \end{bmatrix} = \begin{bmatrix} \boldsymbol{f}^b \\ \boldsymbol{\tau}^b \end{bmatrix}$$

  in which the precise definition of symbols are as above.

Spatial-Frame Lagrangian Derivation

- The kinetic energy T can also be computed by spatial frame velocities:

$$T = \frac{1}{2}m\|\boldsymbol{v}^{s(t)}\|^2 + \frac{1}{2}(\boldsymbol{\omega}^{s(t)})^T \boldsymbol{I}^s \boldsymbol{\omega}^{s(t)}$$

- Force-velocity duality: $\begin{bmatrix} \boldsymbol{f}^{s(t)} \\ \boldsymbol{\tau}^{s(t)} \end{bmatrix}$ and $\begin{bmatrix} \boldsymbol{v}^{s(t)} \\ \boldsymbol{\omega}^{s(t)} \end{bmatrix}$ form a duality pair

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- Justification:
  - The translational kinetic energy can be justified by $\boldsymbol{v}^{s(t)} = R^{s(t)}_{s(t)\to b(t)} \boldsymbol{v}^{b(t)}$
  - The rotational kinetic energy can be justified by $\boldsymbol{\omega}^{s(t)} = R^{s(t)}_{s(t)\to b(t)} \boldsymbol{\omega}^{b(t)}$ and $\boldsymbol{I}^{s(t)} = R^{s(t)}_{s(t)\to b(t)} \boldsymbol{I}^b (R^{s(t)}_{s(t)\to b(t)})^T$
  - The force-velocity duality can be justified by $\boldsymbol{f}^{s(t)} = R^{s(t)}_{s(t)\to b(t)} \boldsymbol{f}^{b(t)}$, $\boldsymbol{\tau}^{s(t)} = R^{s(t)}_{s(t)\to b(t)} \boldsymbol{\tau}^{b(t)}$, and the frame rules for $\boldsymbol{v}$ and $\boldsymbol{\omega}$

Spatial-Frame Lagrangian Derivation

$$\frac{\partial L}{\partial \boldsymbol{v}^{s(t)}} = m\boldsymbol{v}^{s(t)}, \qquad \frac{\partial L}{\partial \boldsymbol{\omega}^{s(t)}} = \boldsymbol{I}^b \boldsymbol{\omega}^{s(t)}$$

- Therefore,

$$\boldsymbol{f}^{s(t_0)} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \boldsymbol{v}^{s(t_0)}} = m\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{v}^{s(t_0)} = m\,\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{v}^{s(t_0)}_{s(t_0)\to b(t)}\Big|_{t=t_0}$$

$$\boldsymbol{\tau}^{s(t_0)} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \boldsymbol{\omega}^{s(t_0)}} = \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{I}^{s(t)}\boldsymbol{\omega}^{s(t_0)}_{s(t_0)\to b(t)}\Big|_{t=t_0} = [\boldsymbol{\omega}^{s(t_0)}_{s(t_0)\to b(t_0)}]\boldsymbol{I}^{s(t_0)}\boldsymbol{\omega}^{s(t_0)}_{s(t_0)\to b(t_0)} + \boldsymbol{I}^{s_0}\,\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\omega}^{s(t)}_{s(t_0)\to b(t)}\Big|_{t=t_0}$$

Spatial-Frame Lagrangian Derivation

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$$\boldsymbol{\tau}^{s(t_0)} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \boldsymbol{\omega}^{s(t_0)}} = \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{I}^{s(t)}\boldsymbol{\omega}^{s(t_0)}_{s(t_0)\to b(t)}\Big|_{t=t_0} = [\boldsymbol{\omega}^{s(t_0)}_{s(t_0)\to b(t_0)}]\boldsymbol{I}^{s(t_0)}\boldsymbol{\omega}^{s(t_0)}_{s(t_0)\to b(t_0)} + \boldsymbol{I}^{s_0}\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\omega}^{s(t)}_{s(t_0)\to b(t)}\Big|_{t=t_0}$$

- In matrix form, we have the famous **spatial frame Newton-Euler equation**:

$$\begin{bmatrix} m\mathrm{Id}_{3\times3} & 0 \\ 0 & \boldsymbol{I}^s \end{bmatrix}\begin{bmatrix} \dot{\boldsymbol{v}}^s \\ \dot{\boldsymbol{\omega}}^s \end{bmatrix} + \begin{bmatrix} 0 \\ \boldsymbol{\omega}^s \times \boldsymbol{I}^s\boldsymbol{\omega}^s \end{bmatrix} = \begin{bmatrix} \boldsymbol{f}^s \\ \boldsymbol{\tau}^s \end{bmatrix}$$

in which the precise definition of symbols are as above.

Example: Robot Arm

Robot Arm

- For kinematic chains with n joints, it is convenient and always possible to choose the joint angles $\theta = (\theta_1, \ldots, \theta_n)$ and the joint torques $\tau = (\tau_1, \ldots, \tau_n)$ as the generalized coordinates and generalized forces, respectively.
  - If joint i is revolute: θ_i joint angle and τ_i is joint torque
  - If joint i is prismatic: θ_i joint position and τ_i is joint force
- Lagrangian Equations:

$$\tau_i = \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{\theta}_i} - \frac{\partial L}{\partial \theta_i}$$

Some Notations

For each link $i = 1, \ldots, n$, $\mathcal{F}_i$ is attached to the center of mass of link i. All the following quantities are expressed in $\mathcal{F}_i$

- $\boldsymbol{\xi}_i^b$: twist of link i
- m_i: mass; $\boldsymbol{I}_i^b$: rotational inertia matrix;
- $\mathfrak{M}_i^b = \begin{bmatrix} m_i \mathrm{Id}_{3\times 3} & 0 \\ 0 & \boldsymbol{I}_i^b \end{bmatrix}$: body inertia matrix
- Kinetic energy of link i: $T_i = \frac{1}{2}(\boldsymbol{\xi}_i^b)^T \mathfrak{M}_i^b \boldsymbol{\xi}_i^b$
- $J_i^b \in \mathbb{R}^{6\times n}$: body Jacobian of link i

Kinetic and Potential Energies

- Total kinetic energy:

$$T(\theta,\dot{\theta}) = \frac{1}{2}\sum_{i=1}^{n}(\boldsymbol{\xi}_i^b)^T\mathfrak{M}_i^b\boldsymbol{\xi}_i^b = \frac{1}{2}\dot{\theta}^T\underbrace{\left(\sum_{i=1}^{n}(J_i^b(\theta)\mathfrak{M}_i^b J_i^b(\theta))\right)}_{\boldsymbol{M}^b(\theta)}\dot{\theta} := \frac{1}{2}\dot{\theta}^T\boldsymbol{M}^b(\theta)\dot{\theta}$$

- Potential energy:

$$V(\theta) = \sum_{i=1}^{n} m_i g h_i(\theta)$$

  - $h_i(\theta)$: height of center of mass of link i

Lagrangian Equation

- Plug $L = T - V$ into $F = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q}$, and we have
- $\tau_i = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta_i}$

$$\tau_i = \sum_{j=1}^{n} M_{ij}^b(\theta)\ddot{\theta}_j + \sum_{j=1}^{n}\sum_{k=1}^{n}\Gamma_{ijk}^b(\theta)\dot{\theta}_j\dot{\theta}_k + \frac{\partial V}{\partial \theta_i}$$

M_{ij}^b is the (i, j)-th entry of matrix $\boldsymbol{M}^b$

- $\Gamma_{ijk}^b(\theta)$ is called the **Christoffel symbols of the first kind**

$$\Gamma_{ijk}^b(\theta) = \frac{1}{2}\left(\frac{\partial M_{ij}^b}{\partial \theta_k} + \frac{\partial M_{ik}^b}{\partial \theta_j} - \frac{\partial M_{jk}^b}{\partial \theta_i}\right)$$

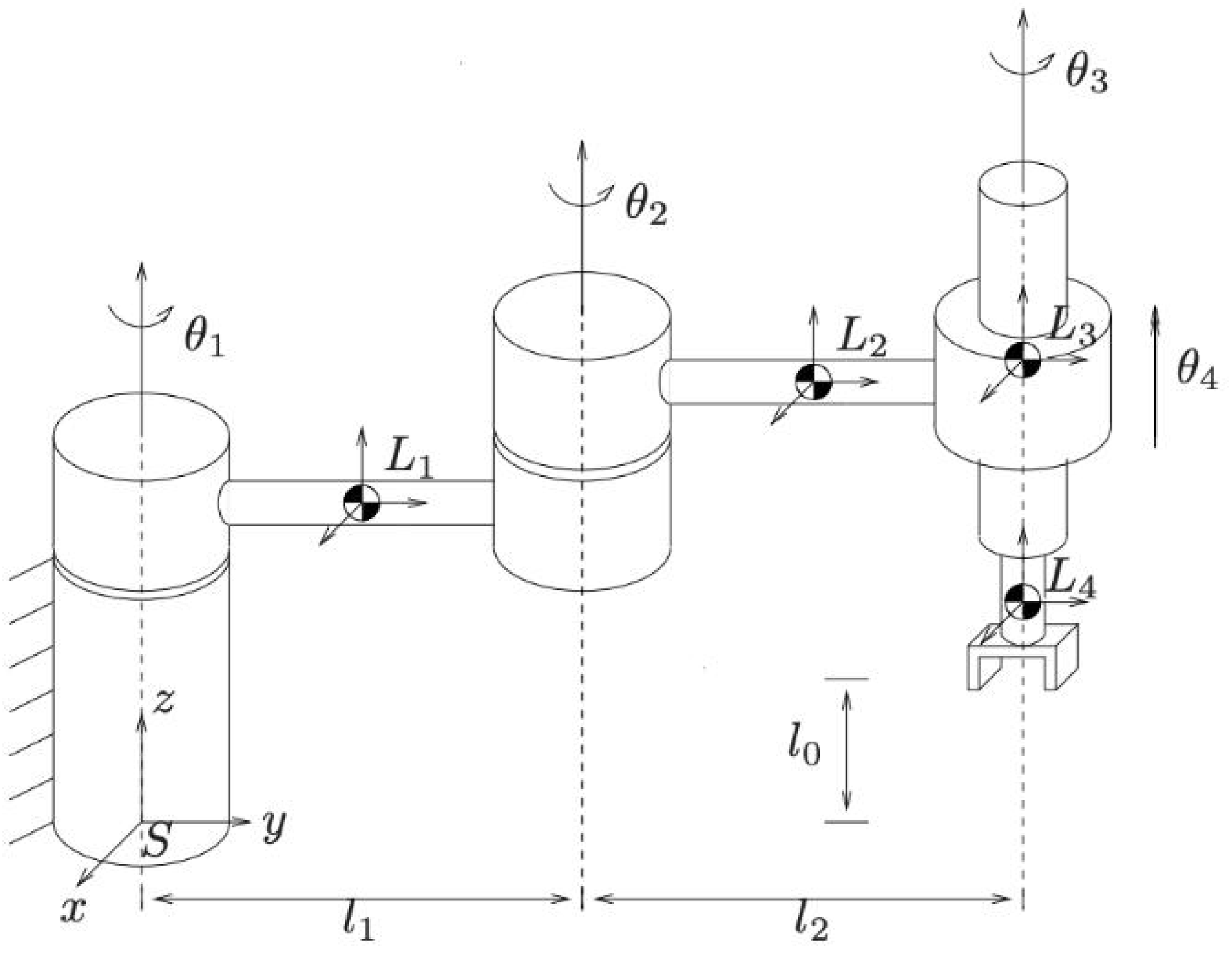
Lagrangian Equation

- Lagrangian equation in vector form:

$$\tau = \boldsymbol{M}^b(\theta)\ddot{\theta} + C^b(\theta, \dot{\theta})\dot{\theta} + g^b(\theta)$$

  - $C^b_{ij}(\theta, \dot{\theta}) := \sum_{k=1}^{n} \Gamma^b_{ijk}\dot{\theta}_k$ is called the **Coriolis matrix**
    - Recall that in the body-frame Newton Euler equation, we also have a Coriolis term that comes from the derivate of rotational inertia. It was used to compensate for the rotational acceleration of the body frame
    - This $C^b_{ij}(\theta, \dot{\theta})$ also comes from taking the derivative of $\boldsymbol{M}^b$ w.r.t. θ. Because $\boldsymbol{M}^b$ and $\boldsymbol{\xi}^b$ are described in the body frame in our derivation, we also need this Coriolis term to compensate for the movement of the body frame.
  - $g^b(\theta)$ is due to gravity in our derivation. If there are other external forces (e.g., friction), it would also show up here.

- Equations for a simple arm



$$
\begin{aligned}
M_{11} &= I_{y2}s_2^2 + I_{y3}s_{23}^2 + I_{z1} + I_{z2}c_2^2 + I_{z3}c_{23}^2 \\
&\quad + m_2 r_1^2 c_2^2 + m_3(l_1c_2 + r_2c_{23})^2 \\
M_{12} &= 0 \\
M_{13} &= 0 \\
M_{21} &= 0 \\
M_{22} &= I_{x2} + I_{x3} + m_3 l_1^2 + m_2 r_1^2 + m_3 r_2^2 + 2m_3 l_1 r_2 c_3 \\
M_{23} &= I_{x3} + m_3 r_2^2 + m_3 l_1 r_2 c_3 \\
M_{31} &= 0 \\
M_{32} &= I_{x3} + m_3 r_2^2 + m_3 l_1 r_2 c_3 \\
M_{33} &= I_{x3} + m_3 r_2^2.
\end{aligned}
$$

$$
\begin{aligned}
\Gamma_{112} &= (I_{y2} - I_{z2} - m_2 r_1^2)c_2 s_2 + (I_{y3} - I_{z3})c_{23}s_{23} \\
&\quad - m_3(l_1c_2 + r_2c_{23})(l_1s_2 + r_2s_{23}) \\
\Gamma_{113} &= (I_{y3} - I_{z3})c_{23}s_{23} - m_3 r_2 s_{23}(l_1c_2 + r_2c_{23}) \\
\Gamma_{121} &= (I_{y2} - I_{z2} - m_2 r_1^2)c_2 s_2 + (I_{y3} - I_{z3})c_{23}s_{23} \\
&\quad - m_3(l_1c_2 + r_2c_{23})(l_1s_2 + r_2s_{23}) \\
\Gamma_{131} &= (I_{y3} - I_{z3})c_{23}s_{23} - m_3 r_2 s_{23}(l_1c_2 + r_2c_{23}) \\
\Gamma_{211} &= (I_{z2} - I_{y2} + m_2 r_1^2)c_2 s_2 + (I_{z3} - I_{y3})c_{23}s_{23} \\
&\quad + m_3(l_1c_2 + r_2c_{23})(l_1s_2 + r_2s_{23}) \\
\Gamma_{223} &= -l_1 m_3 r_2 s_3 \\
\Gamma_{232} &= -l_1 m_3 r_2 s_3 \\
\Gamma_{233} &= -l_1 m_3 r_2 s_3 \\
\Gamma_{311} &= (I_{z3} - I_{y3})c_{23}s_{23} + m_3 r_2 s_{23}(l_1c_2 + r_2c_{23}) \\
\Gamma_{322} &= l_1 m_3 r_2 s_3
\end{aligned}
$$

$$
\begin{bmatrix}
0 \\
-(m_2 g r_1 + m_3 g l_1)\cos\theta_2 - m_3 r_2 \cos(\theta_2 + \theta_3)) \\
-m_3 g r_2 \cos(\theta_2 + \theta_3))
\end{bmatrix}
$$

- Equations for PUMA 560 Arm

$$
\begin{aligned}
I_2 &= I_{zz2} + m_2 * (r_{x2}^2 + r_{y2}^2) + (m_3 + m_4 + m_5 + m_6) * a_2^2 \,; \\
I_3 &= -I_{xx2} + I_{yy2} + (m_3 + m_4 + m_5 + m_6) * a_2^2 \\
&\quad m_2 * r_{x2}^2 - m_2 * r_{y2}^2 \,; \\
I_4 &= m_2 * r_{x2} * (d_2 + r_{z2}) + m_3 * a_2 * r_{z3} \\
&\quad +(m_3 + m_4 + m_5 + m_6) * a_2 * (d_2 + d_3) \,; \\
I_5 &= -m_3 * a_2 * r_{y3} + (m_4 + m_5 + m_6) * a_2 * d_4 + m_4 * a_2 * r_{z4} \,; \\
I_6 &= I_{zz3} + m_3 * r_{y3}^2 + m_4 * a_3^2 + m_4 * (d_4 + r_{z4})^2 + I_{yy4} \\
&\quad +m_5 * a_3^2 + m_5 * d_4^2 + I_{zz5} + m_6 * a_3^2 + m_6 * d_4^2 \\
&\quad +m_6 * r_{z6}^2 + I_{xx6} \,; \\
I_7 &= m_3 * r_{y3}^2 + I_{xx3} - I_{yy3} + m_4 * r_{z4}^2 + 2 * m_4 * d_4 * r_{z4} \\
&\quad +(m_4 + m_5 + m_6) * (d_4^2 - a_3^2) + I_{yy4} - I_{zz4} + I_{zz5} \\
&\quad -I_{yy5} + m_6 * r_{z6}^2 - I_{zz6} + I_{xx6} \,; \\
I_8 &= -m_4 * (d_2 + d_3) * (d_4 + r_{z4}) - (m_5 + m_6) * (d_2 + d_3) * d_4 \\
&\quad m_3 * r_{y3} * r_{z3} + m_3 * (d_2 + d_3) * r_{y3} \,; \\
I_9 &= m_2 * r_{y2} * (d_2 + r_{z2}) \,; \\
I_{10} &= 2 * m_4 * a_3 * r_{z4} + 2 * (m_4 + m_5 + m_6) * a_3 * d_4 \,; \\
I_{11} &= -2 * m_2 * r_{x2} * r_{y2} \,; \\
I_{12} &= (m_4 + m_5 + m_6) * a_2 * a_3 \,; \\
I_{13} &= (m_4 + m_5 + m_6) * a_3 * (d_2 + d_3) \,; \\
I_{14} &= I_{zz4} + I_{yy5} + I_{zz6} \,; \\
I_{15} &= m_6 * d_4 * r_{z6} \,; \\
I_{16} &= m_6 * a_2 * r_{z6} \,; \\
I_{17} &= I_{zz5} + I_{xx6} + m_6 * r_{z6}^2 \,; \\
I_{18} &= m_6 * (d_2 + d_3) * r_{z6} \,; \\
I_{19} &= I_{yy4} - I_{xx4} + I_{zz5} - I_{yy5} + m_6 * r_{z6}^2 + I_{xx6} - I_{zz6} \,; \\
I_{20} &= I_{yy5} - I_{xx5} - m_6 * r_{z6}^2 + I_{zz6} - I_{xx6} \,; \\
I_{21} &= I_{xx4} - I_{yy4} + I_{xx5} - I_{zz5} \,; \\
I_{22} &= m_6 * a_3 * r_{z6} \,; \\
I_{23} &= I_{zz6} \,;
\end{aligned}
$$

Part II. Gravitional Constants

$$
\begin{aligned}
g_1 &= -g * ((m_3 + m_4 + m_5 + m_6) * a_2 + m_2 * r_{x2}) \,; \\
g_2 &= g * (m_3 * r_{y3} - (m_4 + m_5 + m_6) * d_4 - m_4 * r_{z4}) \,; \\
g_3 &= g * m_2 * r_{y2} \,; \\
g_4 &= -g * (m_4 + m_5 + m_6) * a_3 \,; \\
g_5 &= -g * m_6 * r_{z6} \,;
\end{aligned}
$$

Table A3. Computed Values for the Constants Appearing in the Equations of Forces of Motion.

(Inertial constants have units of kilogram meters-squared)

I_1	$=1.43$	± 0.05	I_2	$=1.75$	± 0.07
I_3	$=1.38$	± 0.05	I_4	$=6.90\times10^{-1}$	$\pm 0.20\times10^{-1}$
I_5	$=3.72\times10^{-1}$	$\pm 0.31\times10^{-1}$	I_6	$=3.33\times10^{-1}$	$\pm 0.16\times10^{-1}$
I_7	$=2.98\times10^{-1}$	$\pm 0.29\times10^{-1}$	I_8	$=-1.34\times10^{-1}$	$\pm 0.14\times10^{-1}$
I_9	$=2.38\times10^{-2}$	$\pm 1.20\times10^{-2}$	I_{10}	$=-2.13\times10^{-2}$	$\pm 0.22\times10^{-2}$
I_{11}	$=-1.42\times10^{-2}$	$\pm 0.70\times10^{-2}$	I_{12}	$=-1.10\times10^{-2}$	$\pm 0.11\times10^{-2}$
I_{13}	$=-3.79\times10^{-3}$	$\pm 0.90\times10^{-3}$	I_{14}	$=1.64\times10^{-3}$	$\pm 0.07\times10^{-3}$
I_{15}	$=1.25\times10^{-3}$	$\pm 0.30\times10^{-3}$	I_{16}	$=1.24\times10^{-3}$	$\pm 0.30\times10^{-3}$
I_{17}	$=6.42\times10^{-4}$	$\pm 3.00\times10^{-4}$	I_{18}	$=4.31\times10^{-4}$	$\pm 1.30\times10^{-4}$
I_{19}	$=3.00\times10^{-4}$	$\pm 14.0\times10^{-4}$	I_{20}	$=-2.02\times10^{-4}$	$\pm 8.00\times10^{-4}$
I_{21}	$=-1.00\times10^{-4}$	$\pm 6.00\times10^{-4}$	I_{22}	$=-5.80\times10^{-5}$	$\pm 1.50\times10^{-5}$
I_{23}	$=4.00\times10^{-5}$	$\pm 2.00\times10^{-5}$			
I_{m1}	$=1.14$	± 0.27	I_{m2}	$=4.71$	± 0.54
I_{m3}	$=8.27\times10^{-1}$	$\pm 0.93\times10^{-1}$	I_{m4}	$=2.00\times10^{-1}$	$\pm 0.16\times10^{-1}$
I_{m5}	$=1.79\times10^{-1}$	$\pm 0.14\times10^{-1}$	I_{m6}	$=1.93\times10^{-1}$	$\pm 0.16\times10^{-1}$

(Gravitational constants have units of newton meters)

g_1	$=-37.2$	± 00.5	g_2	$=-8.44$	± 0.20
g_3	$=1.02$	± 0.50	g_4	$=2.49\times10^{-1}$	$\pm 0.25\times10^{-1}$
g_5	$=-2.82\times10^{-2}$	$\pm 0.56\times10^{-2}$			

Table A4. The expressions giving the elements of the kinetic energy matrix.
(The Abbreviated Expressions have units of kg-m².)

$$
\begin{aligned}
a_{11} &= I_{m1} + I_1 + I_3 * CC2 + I_7 * SS23 + I_{10} * SC23 + I_{11} * SC2 \\
&\quad +I_{20} * (SS5 * (SS23 * (1 + CC4) - 1) - 2 * SC23 * C4 * SC5) \\
&\quad +I_{21} * SS23 * CC4 + 2 * \{I_5 * C2 * S23 + I_{12} * C2 * C23 \\
&\quad +I_{15} * (SS23 * C5 + SC23 * C4 * S5) \\
&\quad +I_{16} * C2 * (S23 * C5 + C23 * C4 * S5) \\
&\quad +I_{18} * S4 * S5 + I_{22} * (SC23 * C5 + CC23 * C4 * S5)\} \,; \\
&\approx 2.57 + 1.38 * CC2 + 0.30 * SS23 + 7.44\times10^{-1} * C2 * S23 \,. \\
a_{12} &= I_4 * S2 + I_8 * C23 + I_9 * C2 + I_{13} * S23 - I_{15} * C23 * S4 * S5 \\
&\quad +I_{16} * S2 * S4 * S5 + I_{18} * (S23 * C4 * S5 - C23 * C5) \\
&\quad +I_{19} * S23 * SC4 + I_{20} * S4 * (S23 * C4 * CC5 + C23 * SC5) \\
&\quad +I_{22} * S23 * S4 * S5 \,; \\
&\approx 6.90\times10^{-1} * S2 - 1.34\times10^{-1} * C23 + 2.38\times10^{-2} * C2 \,. \\
a_{13} &= I_8 * C23 + I_{13} * S23 - I_{15} * C23 * S4 * S5 + I_{19} * S23 * SC4 \\
&\quad +I_{18} * (S23 * C4 * S5 - C23 * C5) + I_{22} * S23 * S4 * S5 \\
&\quad +I_{20} * S4 * (S23 * C4 * CC5 + C23 * SC5) \,; \\
&\approx -1.34\times10^{-1} * C23 + -3.97\times10^{-3} * S23 \,. \\
a_{14} &= I_{14} * C23 + I_{15} * S23 * C4 * S5 + I_{16} * C2 * C4 * S5 \\
&\quad +I_{18} * C23 * S4 * S5 - I_{20} * (S23 * C4 * SC5 + C23 * SS5) \\
&\quad +I_{22} * C23 * C4 * S5 \,; \qquad \approx 0 \,. \\
a_{15} &= I_{15} * S23 * S4 * C5 + I_{16} * C2 * S4 * C5 + I_{17} * S23 * S4 \\
&\quad +I_{18} * (S23 * S5 - C23 * C4 * C5) + I_{22} * C23 * S4 * C5 \,; \\
&\approx 0 \,. \\
a_{16} &= I_{23} * (C23 * C5 - S23 * C4 * S5) \,; \qquad \approx 0 \,. \\
a_{22} &= I_{m2} + I_2 + I_6 + I_{20} * SS4 * SS5 + I_{21} * SS4 \\
&\quad +2 * \{I_5 * S3 + I_{12} * C3 + I_{15} * C5 \\
&\quad +I_{16} * (S3 * C5 + C3 * C4 * S5) + I_{22} * C4 * S5\} \,; \\
&\approx 6.79 + 7.44\times10^{-1} * S3 \,. \\
a_{23} &= I_5 * S3 + I_6 + I_{12} * C3 + I_{16} * (S3 * C5 + C3 * C4 * S5) \\
&\quad +I_{20} * SS4 * SS5 + I_{21} * SS4 + 2 * \{I_{15} * C5 + I_{22} * C4 * S5\} \,; \\
&\approx .333 + 3.72\times10^{-1} * S3 - 1.10\times10^{-2} * C3 \,. \\
a_{24} &= -I_{15} * S4 * S5 - I_{16} * S3 * S4 * S5 + I_{20} * S4 * SC5 \,; \\
&\approx 0 \,. \\
a_{25} &= I_{15} * C4 * C5 + I_{16} * (C3 * S5 + S3 * C4 * C5) \\
&\quad +I_{17} * C4 + I_{22} * S5 \,; \qquad \approx 0 \,. \\
a_{26} &= I_{23} * S4 * S5 \,; \qquad \approx 0 \,. \\
a_{33} &= I_{m3} + I_6 + I_{20} * SS4 * SS5 + I_{21} * SS4 \\
&\quad +2 * \{I_{15} * C5 + I_{22} * C4 * S5\} \,; \qquad \approx 1.16 \,. \\
a_{34} &= -I_{15} * S4 * S5 + I_{20} * S4 * SC5 \,; \\
&\approx -1.25\times10^{-3} * S4 * S5 \,. \\
a_{35} &= I_{15} * C4 * C5 + I_{17} * C4 + I_{22} * S5 \,; \\
&\approx 1.25\times10^{-3} * C4 * C5 \,. \\
a_{36} &= I_{23} * S4 * S5 \,; \qquad \approx 0 \,. \\
a_{44} &= I_{m4} + I_{14} - I_{20} * SS5 \,; \qquad \approx 0.20 \,. \\
a_{45} &= 0. \\
a_{46} &= I_{23} * C5 \,; \qquad \approx 0 \,. \\
a_{55} &= I_{m5} + I_{17} \,; \qquad \approx 0.18 \,. \\
a_{56} &= 0. \\
a_{66} &= I_{m6} + I_{23} \,; \qquad \approx 0.19 \,.
\end{aligned}
$$

Table A5. The expressions giving the elements of the Coriolis matrix.
(The Abbreviated Expressions have units of kg-m².)

$$
\begin{aligned}
b_{112} &= 2 * \{-I_3 * SC2 + I_5 * C223 + I_7 * SC23 - I_{12} * S223 \\
&\quad +I_{15} * (2 * SC23 * C5 + (1 - 2 * SS23) * C4 * S5) \\
&\quad +I_{16} * (C223 * C5 - S223 * C4 * S5) + I_{21} * SC23 * CC4 \\
&\quad +I_{20} * ((1 + CC4) * SC23 * SS5 - (1 - 2 * SS23) * C4 * SC5) \\
&\quad +I_{22} * ((1 - 2 * SS23) * C5 - 2 * SC23 * C4 * S5)\} \\
&\quad +I_{10} * (1 - 2 * SS23) + I_{11} * (1 - 2 * SS2) \,; \\
&\approx -2.76 * SC2 + 7.44\times10^{-1} * C223 + 0.60 * SC23 \\
&\quad - 2.13\times10^{-2} * (1 - 2 * SS23) \,. \\
b_{113} &= 2 * \{I_5 * C2 * C23 + I_7 * SC23 - I_{12} * C2 * S23 \\
&\quad +I_{15} * (2 * SC23 * C5 + (1 - 2 * SS23) * C4 * S5) \\
&\quad +I_{16} * C2 * (C23 * C5 - S23 * C4 * S5) + I_{21} * SC23 * CC4 \\
&\quad +I_{20} * ((1 + CC4) * SC23 * SS5 - (1 - 2 * SS23) * C4 * SC5) \\
&\quad +I_{22} * ((1 - 2 * SS23) * C5 - 2 * SC23 * C4 * S5)\} \\
&\quad +I_{10} * (1 - 2 * SS23) \,; \\
&\approx 7.44\times10^{-1} * C2 * C23 + 0.60 * SC23 \\
&\quad + 2.20\times10^{-2} * C2 * S23 - 2.13\times10^{-2} * (1 - 2 * SS23) \,. \\
b_{114} &= 2 * \{-I_{15} * SC23 * S4 * S5 - I_{16} * C2 * C23 * S4 * S5 \\
&\quad +I_{18} * C4 * S5 - I_{20} * (SS23 * SS5 * SC4 - SC23 * S4 * SC5) \\
&\quad -I_{22} * CC23 * S4 * S5 - I_{21} * SS23 * SC4\} \,; \\
&\approx -2.50\times10^{-3} * SC23 * S4 * S5 + 8.60\times10^{-4} * C4 * S5 \\
&\quad - 2.48\times10^{-3} * C2 * C23 * S4 * S5 \,. \\
b_{115} &= 2 * \{I_{20} * (SC5 * (CC4 * (1 - CC23) - CC23) \\
&\quad -SC23 * C4 * (1 - 2 * SS5)) - I_{15} * (SS23 * S5 - SC23 * C4 * C5) \\
&\quad -I_{16} * C2 * (S23 * S5 - C23 * C4 * C5) + I_{18} * S4 * C5 \\
&\quad +I_{22} * (CC23 * C4 * C5 - SC23 * S5)\} \,; \\
&\approx -2.50\times10^{-3} * (SS23 * S5 - SC23 * C4 * C5) \\
&\quad - 2.48\times10^{-3} * C2 * (S23 * S5 - C23 * C4 * C5) \\
&\quad + 8.60\times10^{-4} * S4 * C5. \\
b_{116} &= 0 \,. \\
b_{123} &= 2 * \{-I_8 * S23 + I_{13} * C23 + I_{15} * S23 * S4 * S5 \\
&\quad +I_{18} * (C23 * C4 * S5 + S23 * C5) + I_{19} * C23 * SC4 \\
&\quad +I_{20} * S4 * (C23 * C4 * CC5 - S23 * SC5) \\
&\quad +I_{22} * C23 * S4 * S5\} \,; \\
&\approx 2.67\times10^{-1} * S23 - 7.58\times10^{-3} * C23 \,. \\
b_{124} &= -I_{18} * 2 * S23 * S4 * S5 + I_{19} * S23 * (1 - (2 * SS4)) \\
&\quad +I_{20} * S23 * (1 - 2 * SS4 * CC5) - I_{14} * S23 \,; \qquad \approx 0 \,. \\
b_{125} &= I_{17} * C23 * S4 + I_{18} * 2 * (S23 * C4 * C5 + C23 * S5) \\
&\quad +I_{20} * S4 * (C23 * (1 - 2 * SS5) - S23 * C4 * 2 * SC5) \,; \\
&\approx 0 \,. \\
b_{126} &= -I_{23} * (S23 * C5 + C23 * C4 * S5) \,; \qquad \approx 0 \,. \\
b_{134} &= b_{124} \,. \qquad b_{135} = b_{125} \,. \qquad b_{136} = b_{126} \,. \\
b_{145} &= 2 * \{I_{15} * S23 * C4 * C5 + I_{16} * C2 * C4 * C5 \\
&\quad +I_{18} * C23 * S4 * C5 + I_{22} * C23 * C4 * C5\} + I_{17} * S23 * C4 \\
&\quad -I_{20} * (S23 * C4 * (1 - 2 * SS5) + 2 * C23 * SC5) \,; \\
&\approx 0 \,. \\
b_{146} &= I_{23} * S23 * S4 * S5 \,; \qquad \approx 0 \,. \\
b_{156} &= -I_{23} * (C23 * S5 + S23 * C4 * C5) \,; \qquad \approx 0 \,. \\
b_{212} &= 0 \,. \qquad b_{213} = 0 \,. \\
b_{214} &= I_{14} * S23 + I_{19} * S23 * (1 - (2 * SS4)) \\
&\quad +2 * \{-I_{15} * C23 * C4 * S5 + I_{16} * S2 * C4 * S5 \\
&\quad +I_{20} * (S23 * (CC5 * CC4 - 0.5) + C23 * C4 * SC5) \\
&\quad +I_{22} * S23 * C4 * S5\} \,; \\
&\approx 1.64\times10^{-3} * S23 - 2.50\times10^{-3} * C23 * C4 * S5 + \\
&\quad 2.48\times10^{-3} * S2 * C4 * S5 + 0.30\times10^{-3} * S23 * (1 - (2 * SS4)) \,. \\
b_{215} &= 2 * \{-I_{15} * C23 * S4 * C5 + I_{22} * S23 * S4 * C5 \\
&\quad +I_{16} * S2 * S4 * C5\} - I_{17} * C23 * S4 \\
&\quad +I_{20} * (C23 * S4 * (1 - 2 * SS5) - 2 * S23 * SC4 * SC5) \,; \\
&\approx -2.50\times10^{-3} * C23 * S4 * C5 + 2.48\times10^{-3} * S2 * S4 * C5 \\
&\quad - 6.42\times10^{-4} * C23 * S4 \,. \\
b_{216} &= -b_{126} \,. \\
b_{223} &= 2 * \{-I_{12} * S3 + I_5 * C3 + I_{16} * (C3 * C5 - S3 * C4 * S5)\} \,; \\
&\approx 2.20\times10^{-2} * S3 + 7.44\times10^{-1} * C3 \,. \\
b_{224} &= 2 * \{-I_{16} * C3 * S4 * S5 + I_{20} * SC4 * SS5 \\
&\quad +I_{21} * SC4 - I_{22} * S4 * S5\} \,; \\
&\approx -2.48\times10^{-3} * C3 * S4 * S5 \,. \\
b_{225} &= 2 * \{-I_{15} * S5 + I_{16} * (C3 * C4 * C5 - S3 * S5) \\
&\quad +I_{20} * SS4 * SC5 + I_{22} * C4 * C5\} \,; \\
&\approx -2.50\times10^{-3} * S5 + 2.48\times10^{-3} * (C3 * C4 * C5 - S3 * S5) \,. \\
b_{226} &= 0 \,. \qquad b_{234} = b_{224} \,. \\
b_{235} &= b_{225} \,. \qquad b_{236} = 0 \,. \\
b_{245} &= 2 * \{-I_{15} * S4 * C5 - I_{16} * S3 * S4 * C5\} \\
&\quad -I_{17} * S4 + I_{20} * S4 * (1 - 2 * SS5) \,; \qquad \approx 0 \,. \\
b_{246} &= I_{23} * C4 * S5 \,; \qquad \approx 0 \,. \\
b_{256} &= I_{23} * S4 * C5 \,; \qquad \approx 0 \,. \\
b_{312} &= 0 \,. \qquad b_{313} = 0 \,. \\
b_{314} &= 2 * \{-I_{15} * C23 * C4 * S5 + I_{22} * S23 * C4 * S5 \\
&\quad +I_{20} * (S23 * (CC5 * CC4 - 0.5) + C23 * C4 * SC5)\} \\
&\quad +I_{14} * S23 + I_{19} * S23 * (1 - (2 * SS4)) \,; \\
&\approx -2.50\times10^{-3} * C23 * C4 * S5 + 1.64\times10^{-3} * S23 \\
&\quad + 0.30\times10^{-3} * S23 * (1 - 2 * SS4) \,. \\
b_{315} &= 2 * \{-I_{15} * C23 * S4 * C5 + I_{22} * S23 * S4 * C5\} \\
&\quad -I_{17} * C23 * S4 \\
&\quad +I_{20} * S4 * (C23 * (1 - 2 * SS5) - 2 * S23 * C4 * SC5) \,; \\
&\approx -2.50\times10^{-3} * C23 * S4 * C5 - 6.42\times10^{-4} * C23 * S4 \,. \\
b_{316} &= -b_{136} \,. \qquad b_{323} = 0 \,. \\
b_{324} &= 2 * \{I_{20} * SC4 * SS5 + I_{21} * SC4 - I_{22} * S4 * S5\} \,; \\
&\approx 0 \,. \\
b_{325} &= 2 * \{-I_{15} * S5 + I_{20} * SS4 * SC5 + I_{22} * C4 * C5\} \,; \\
&\approx -2.50\times10^{-3} * S5 \,. \\
b_{326} &= 0 \,. \qquad b_{334} = b_{324} \,. \\
b_{335} &= b_{325} \,. \qquad b_{336} = 0 \,. \\
b_{345} &= -I_{15} * 2 * S4 * C5 - I_{17} * S4 + I_{20} * S4 * (1 - 2 * SS5) \,; \\
&\approx -2.50\times10^{-3} * S4 * C5 \,. \\
b_{346} &= b_{246} \,. \qquad b_{356} = b_{256} \,. \qquad b_{412} = -b_{214} \,. \\
b_{413} &= -b_{314} \,. \qquad b_{414} = 0 \,. \\
b_{415} &= -I_{20} * (S23 * C4 * (1 - 2 * SS5) + 2 * C23 * SC5) \\
&\quad -I_{17} * S23 * C4 \,; \\
&\approx -6.42\times10^{-4} * S23 * C4 \,. \\
b_{416} &= -b_{146} \,. \qquad b_{423} = -b_{324} \,. \qquad b_{424} = 0 \,. \\
b_{425} &= I_{17} * S4 + I_{20} * S4 * (1 - 2 * SS5) \,; \\
&\approx 6.42\times10^{-4} * S4 \,. \\
b_{426} &= -b_{246} \,. \qquad b_{434} = 0 \,. \\
b_{435} &= b_{425} \,. \qquad b_{436} = -b_{346} \,. \\
b_{445} &= -I_{20} * 2 * SC5 \,; \qquad \approx 0 \,. \\
b_{446} &= 0 \,; \\
b_{456} &= -I_{23} * S5 \,; \qquad \approx 0 \,. \\
b_{512} &= -b_{215} \,. \qquad b_{513} = -b_{315} \,. \qquad b_{514} = -b_{415} \,. \\
b_{515} &= 0 \,. \qquad b_{516} = -b_{156} \,. \qquad b_{523} = -b_{325} \,. \\
b_{524} &= -b_{425} \,. \qquad b_{525} = 0 \,. \qquad b_{526} = -b_{256} \,. \\
b_{534} &= b_{524} \,. \qquad b_{535} = 0 \,. \qquad b_{536} = -b_{356} \,. \\
b_{545} &= 0 \,. \qquad b_{546} = -b_{456} \,. \qquad b_{556} = 0 \,. \\
b_{612} &= b_{126} \,. \qquad b_{613} = b_{136} \,. \qquad b_{614} = b_{146} \,. \\
b_{615} &= b_{156} \,. \qquad b_{616} = 0 \,. \qquad b_{623} = 0 \,. \\
b_{624} &= b_{246} \,. \qquad b_{625} = b_{256} \,. \qquad b_{626} = 0 \,. \\
b_{634} &= b_{624} \,. \qquad b_{635} = b_{625} \,. \qquad b_{636} = 0 \,. \\
b_{645} &= b_{456} \,. \qquad b_{646} = 0 \,. \qquad b_{656} = 0 \,.
\end{aligned}
$$

Table A6. The expressions for the terms of the centrifugal matrix.
(The Abbreviated Expressions have units of kg-m².)

$$
\begin{aligned}
c_{11} &= 0 \,. \\
c_{12} &= +I4 * C2 - I8 * S23 - I9 * S2 + I13 * C23 \\
&\quad +I_{15} * S23 * S4 * S5 + I_{16} * C2 * S4 * S5 \\
&\quad +I_{18} * (C23 * C4 * S5 + S23 * C5) + I_{19} * C23 * SC4 \\
&\quad +I_{20} * S4 * (C23 * C4 * cC5 - S23 * SC5) \\
&\quad +I_{22} * C23 * S4 * S5 \,; \\
&\approx 6.90\times10^{-1} * C2 + 1.34\times10^{-1} * S23 - 2.38\times10^{-2} * S2 \,. \\
c_{13} &= 0.5 * b_{123} \,. \\
c_{14} &= -I_{15} * S23 * S4 * S5 - I_{16} * C2 * S4 * S5 \\
&\quad +I_{18} * C23 * C4 * S5 + I_{20} * S23 * S4 * SC5 \\
&\quad -I_{22} * C23 * S4 * S5 \,; \qquad \approx 0 \,. \\
c_{15} &= -I_{15} * S23 * S4 * S5 - I_{16} * C2 * S4 * S5 \,; \\
&\quad +I_{18} * (S23 * C5 + C23 * C4 * S5) - I_{22} * C23 * S4 * S5 \\
&\approx 0 \,. \\
c_{16} &= 0 \,. \qquad c_{21} = -0.5 * b_{112} \,. \\
c_{22} &= 0 \,. \qquad c_{23} = 0.5 * b_{223} \,. \\
c_{24} &= -I_{15} * C4 * S5 - I_{16} * S3 * C4 * S5 + I_{20} * C4 * SC5 \,; \\
&\approx 0 \,. \\
c_{25} &= -I_{15} * C4 * S5 + I_{16} * (C3 * C5 - S3 * C4 * S5) \\
&\quad +I_{22} * C5 \,; \qquad \approx 0 \,. \\
c_{26} &= 0 \,. \qquad c_{31} = -0.5 * b_{113} \,. \\
c_{32} &= -c_{23} \,. \qquad c_{33} = 0 \,. \\
c_{34} &= -I_{15} * C4 * S5 + I_{20} * C4 * SC5 \,; \\
&\approx -1.25\times10^{-3} * C4 * S5 \,. \\
c_{35} &= -I_{15} * C4 * S5 + I_{22} * C5 \,; \qquad \approx c_{34} \,. \\
c_{36} &= 0 \,. \qquad c_{41} = -0.5 * b_{114} \,. \qquad c_{42} = -0.5 * b_{224} \\
c_{43} &= 0.5 * b_{423} \,. \qquad c_{44} = 0 \,. \qquad c_{45} = 0 \,. \\
c_{46} &= 0 \,. \qquad c_{51} = -0.5 * b_{115} \,. \qquad c_{52} = -0.5 * b_{225} \\
c_{53} &= 0.5 * b_{523} \,. \qquad c_{54} = -0.5 * b_{445} \,. \qquad c_{55} = 0 \,. \\
c_{56} &= 0 \,. \qquad c_{61} = 0 \,. \qquad c_{62} = 0 \,. \\
c_{63} &= 0 \,. \qquad c_{64} = 0 \,. \qquad c_{65} = 0 \,. \\
c_{66} &= 0 \,.
\end{aligned}
$$

Table A7. Gravity Terms.
(The Abbreviated Expressions have units of newton-meters.)

$$
\begin{aligned}
g_1 &= 0 \,. \\
g_2 &= g1 * C2 + g2 * S23 + g3 * S2 + g4 * C23 \,; \\
&\quad +g5 * (S23 * C5 + C23 * C4 * S5) \\
&\approx -37.2 * C2 - 8.4 * S23 + 1.02 * S2 \,. \\
g_3 &= g2 * S23 + g4 * C23 + g5 * (S23 * C5 + C23 * C4 * S5) \,; \\
&\approx -8.4 * S23 + 0.25 * C23 \,. \\
g_4 &= -g5 * S23 * S4 * S5 \,; \\
&\approx 2.8\times10^{-2} * S23 * S4 * S5 \,. \\
g_5 &= g5 * (C23 * S5 + S23 * C4 * C5) \,; \\
&\approx -2.8\times10^{-2} * (C23 * S5 + S23 * C4 * C5) \,. \\
g_6 &= 0 \,.
\end{aligned}
$$

L9: Lagrangian Dynamics

Agenda

Dynamics Example: Grasp

Dynamics Example: Grasp

Lagrangian vs. Newton-Euler Methods

Lagrangian Method

Generalized Coordinates and Forces

Generalized Coordinates and Forces

L9: Lagrangian Dynamics

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The flow and some contents are based on ECE5463 taught at Ohio State University by Prof. Wei Zhang