# L8: Basic Concepts of Rigid-Body Dynamics

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# Some Further Clarifications on Velocity

- Consider we observe the motion of a moving body $b(t)$ from a moving observer's frame $o(t)$, and the recording of motion is relative to $s(t)$.
- **velocity** observed from an arbitrary frame (e.g., point linear velocity, angular velocity, and twist) is:

$$\boldsymbol{v}_{s(t_0)\to b(t_0)}^{o(t_0)} = \frac{\mathrm{d}}{\mathrm{d}t} p_{s(t)\to b(t)}^{o(t_0)} \bigg|_{t=t_0}, \qquad \text{where } p_{s(t)\to b(t)}^{o(t_0)} = p_{o(t_0)\to b(t)}^{o(t_0)} - p_{o(t_0)\to s(t)}^{o(t_0)}$$

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- Note: above is a general rule of **taking derivative of coordinate w.r.t. time** (so that body velocity/twist is non-zero).
  - e.g., when we derive the body-frame Euler equation last lecture, body inertia is defined by body frame coordinates, more precisely, $I_{b(t)}^{b(t_0)} = \int \mathrm{d}V \rho(r_{b(t)\to x(t)}^{b(t_0)})[r_{b(t)\to x(t)}^{b(t_0)}][r_{b(t)\to x(t)}^{b(t_0)}]$
  - Therefore, while $I$ is an invariant w.r.t. $t$ (since $r_{b(t)\to x(t)}^{b(t_0)}\Big|_{t=t_0}$ is invariant), its derivative (taken only w.r.t. the subscript) is non-zero.
  - This is exactly the case that body-frame coordinate is a constant for body points, but they have velocity.

# Agenda

- Kinetic Energy
- Change of Frame for Various Quantities
- Forward and Inverse Dynamics

click to jump to the section.

# Kinetic Energy

# Kinetic Energy for Point Mass

- If a point mass $m$ is moving with velocity $\boldsymbol{v}^o_{s(t)\to b(t)}$ ($s(t)$ is an inertia frame and the origin of $b(t)$ is the point), then the kinetic energy of the point mass is

$$T_{s(t)\to b(t)} = \frac{1}{2} m \|\boldsymbol{v}^o_{s(t)\to b(t)}\|^2 \qquad \text{(kinetic energy)}$$

# Observer-Independence of Kinetic Energy

- Note that we omit observer's frame when describing kinetic energy, because it is independent of the observer's frame.
- We prove by showing that $\|\boldsymbol{v}^{o_1}_{s(t)\to b(t)}\| = \|\boldsymbol{v}^{o_2}_{s(t)\to b(t)}\|$

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- Proof:

$$\begin{aligned}
\boldsymbol{v}^{o_1}_{s(t)\to b(t)} &= \boldsymbol{v}^{o_1}_{b(t)} - \boldsymbol{v}^{o_1}_{s(t)} = [\boldsymbol{\xi}^{o_1}_{b(t)} - \boldsymbol{\xi}^{o_1}_{s(t)}]p^{o_1} = T^{-1}_{o_2\to o_1}[\boldsymbol{\xi}^{o_2}_{b(t)} - \boldsymbol{\xi}^{o_2}_{s(t)}]T_{o_2\to o_1}p^{o_1} \\
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$$\therefore \|\boldsymbol{v}^{o_1}_{s(t)\to b(t)}\| = \|\boldsymbol{v}^{o_2}_{s(t)\to b(t)}\|$$

- We have also derived

$$\boldsymbol{v}^{o_1}_{s(t)\to b(t)} = R_{o_1\to o_2}\boldsymbol{v}^{o_2}_{s(t)\to b(t)} \text{ for } \boldsymbol{v} \in \mathbb{R}^3 \quad \text{(change of frame for velocities)}$$

# Kinetic Energy for Rigid Body

- Integrate kinetic energy of every point mass over the body
- We choose the body frame $\mathcal{F}_{b(t)}$ to start the derivation. Using the independence of observer's frame, we derive the formula to compute the energy in other frames.
- *The origin of our body frame is always at the center of mass of the body*

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- Integrate kinetic energy of every point mass over the body
- We choose the body frame $\mathcal{F}_{b(t)}$ to start the derivation. Using the independence of observer's frame, we derive the formula to compute the energy in other frames.
- *The origin of our body frame is always at the center of mass of the body*
- The velocity of a body point $\boldsymbol{r}^{b(t)}$ is

$$\begin{aligned}\boldsymbol{v}_{s(t)\to b(t)}^{b(t)} &= [\boldsymbol{\xi}_{s(t)\to b(t)}^{b(t)}]\boldsymbol{r}^{b(t)} \\ &= \begin{bmatrix} [\boldsymbol{\omega}_{s(t)\to b(t)}^{b(t)}] & \dot{\boldsymbol{t}}_{s(t)\to b(t)}^{b(t)} \\ 0 & 0 \end{bmatrix}\boldsymbol{r}^{b(t)}\end{aligned}$$

# Kinetic Energy for Rigid Body

Therefore,

$$\begin{aligned} T_{s(t)\to b(t)} &= \int_{\boldsymbol{r}^b\in B} \frac{1}{2}\rho(x)\mathrm{d}V\|\boldsymbol{v}^{b(t)}_{s(t)\to b(t)}\|^2 = \int_{\boldsymbol{r}^b\in B} \frac{1}{2}\rho(x)\mathrm{d}V\|[\boldsymbol{\omega}^{b(t)}_{s(t)\to b(t)}]\boldsymbol{r}^{b(t)} + \dot{\boldsymbol{t}}^{b(t)}_{s(t)\to b(t)}\|^2 \\ &= (\text{some derivations using } [\boldsymbol{\omega}]\boldsymbol{r} = -[\boldsymbol{r}]\boldsymbol{\omega}) \\ &= \frac{1}{2}m\|\dot{\boldsymbol{t}}^{b(t)}_{s(t)\to b(t)}\|^2 + \frac{1}{2}(\boldsymbol{\omega}^{b(t)}_{s(t)\to b(t)})^T\boldsymbol{I}^b\boldsymbol{\omega}^{b(t)}_{s(t)\to b(t)} \\ &= \frac{1}{2}(\boldsymbol{\xi}^{b(t)}_{s(t)\to b(t)})^T\mathfrak{M}^b\boldsymbol{\xi}^{b(t)}_{s(t)\to b(t)} \end{aligned}$$

where

$$\mathfrak{M}^b = \begin{bmatrix} m\mathrm{Id}_{3\times 3} & 0 \\ 0 & \boldsymbol{I}^b \end{bmatrix} \in \mathbb{R}^{6\times 6}$$

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  - Recall that $\boldsymbol{\xi}^1 = [\mathrm{Ad}_{1\to 2}]\boldsymbol{\xi}^2$, and we conclude that

$$\mathfrak{M}^2 = [\mathrm{Ad}_{1\to 2}]^{\mathrm{T}}\mathfrak{M}^1[\mathrm{Ad}_{1\to 2}] \qquad \text{(change of frame)}$$

# Change of Observer's Frame for Rotational Inertia Matrix

- A side-product of introducing $\mathfrak{M}^o$ is that we can compute the inertia matrix in other frames conveniently
- We derived the change of frame formula for different body frames. **What about frame change between general observer's frames?**

# Change of Observer's Frame for Rotational Inertia Matrix

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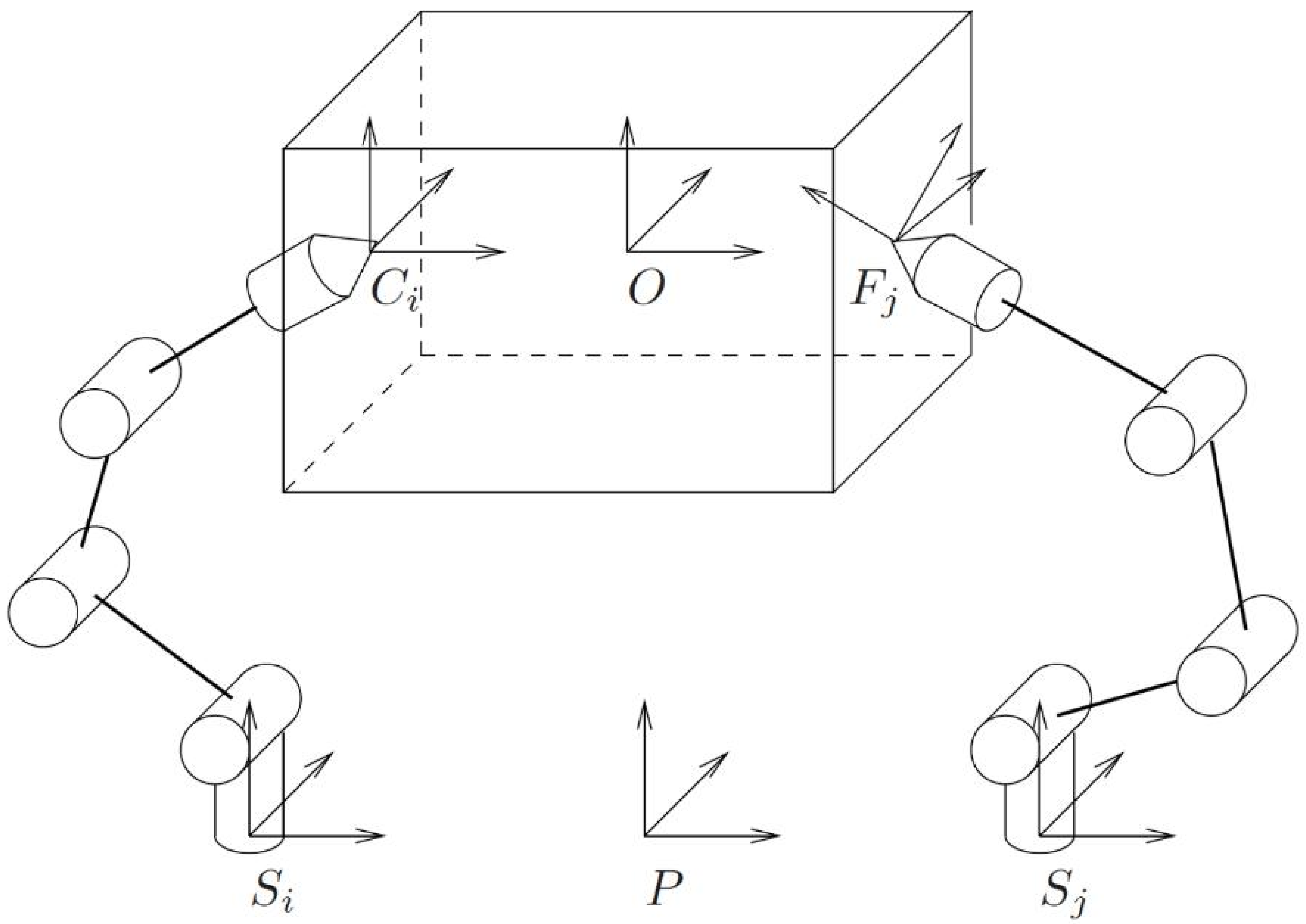
- We derived the change of frame formula for different body frames. **What about frame change between general observer's frames?**
- One can verify that,

$$\textbf{the bottom-right } 3 \times 3 \textbf{ block of } \mathfrak{M}^2 \textbf{ is the rotational inertial matrix in } \mathcal{F}_2$$

# Change of Frame for Various Quantities

# Motivating Example: Grasp Problem

- Consider the right grasp problem

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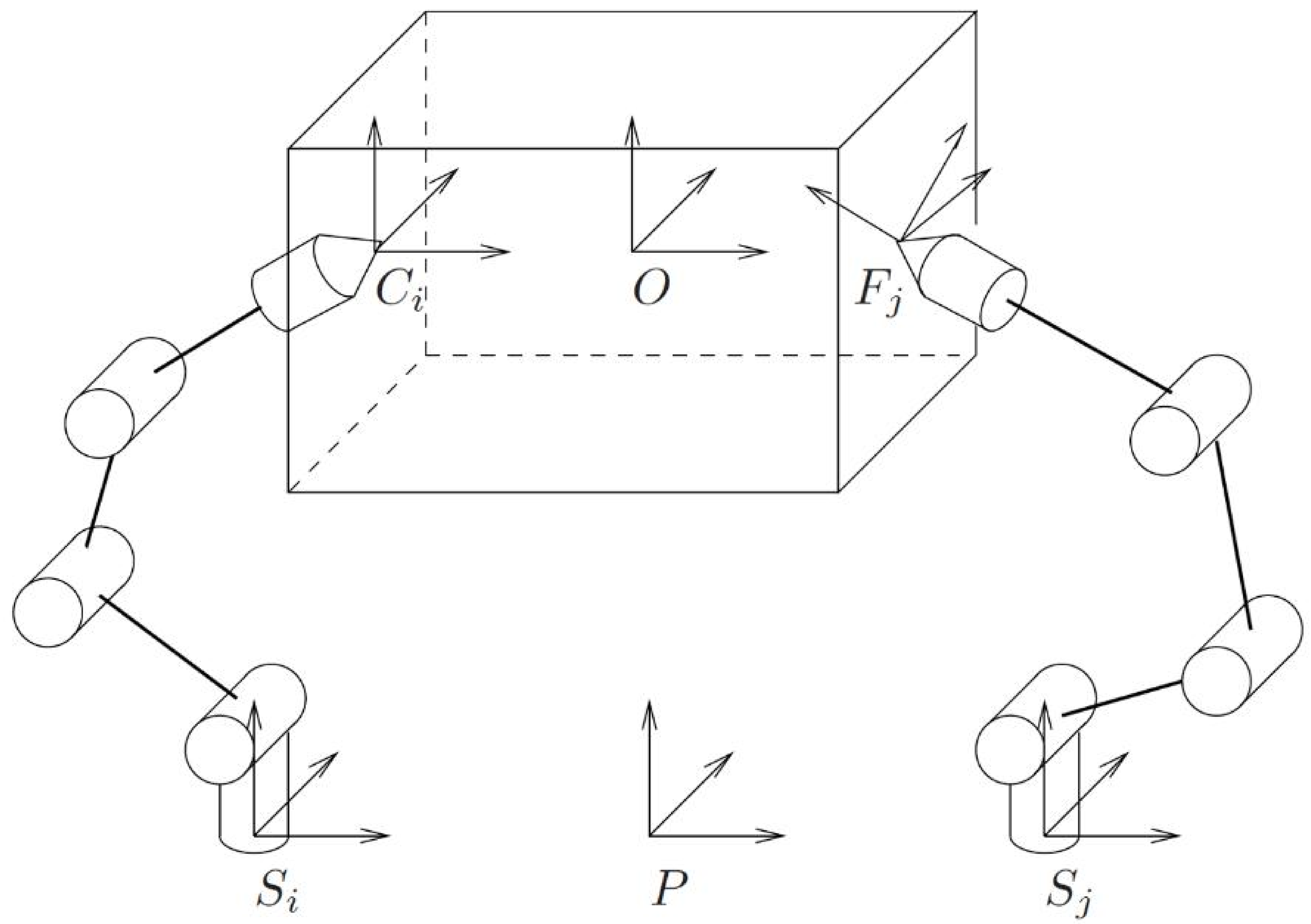


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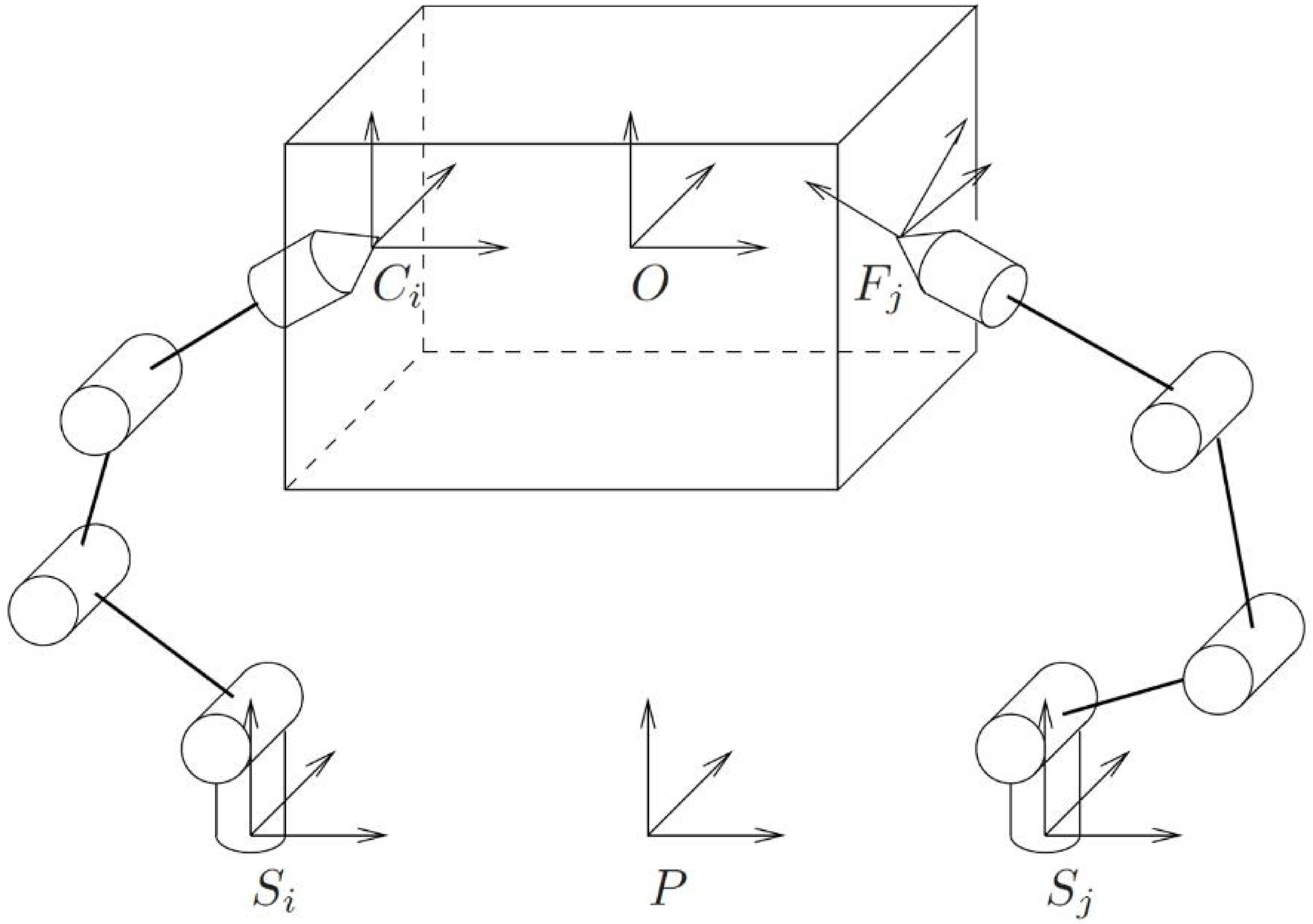


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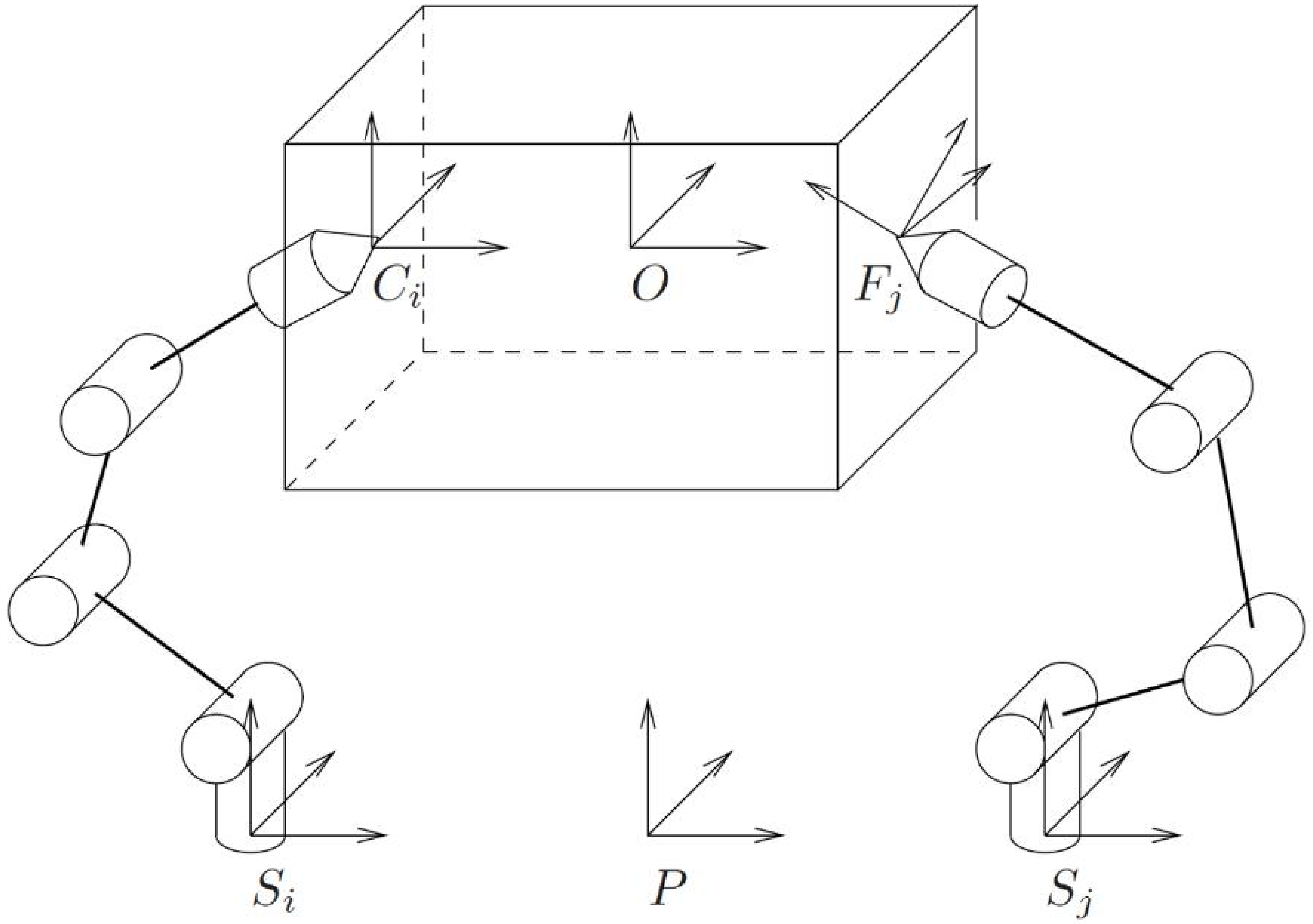


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  - Assume that we are grasping this box using two arms
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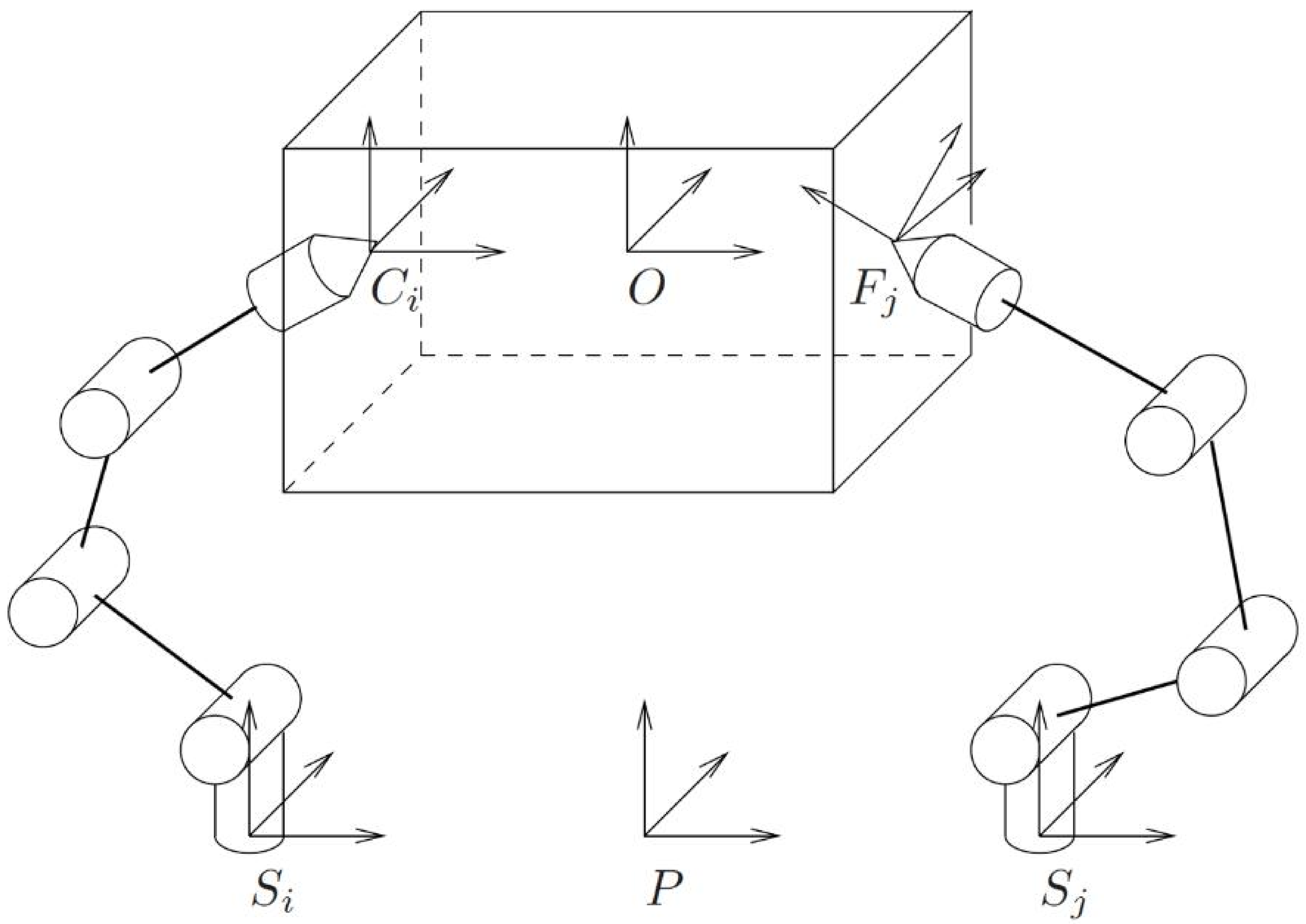
  - The contact area will create certain force and torque at the contact point
    - force: pressure and friction
    - torque: e.g., anti-twisting friction force caused by the area contact

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# Contact Coordinate Frame

- We build a **contact frame** $C_i$ at each contact point
- The $z$-axis of the frame points inward along *surface normal*
- When recording force and torque at the contact point, it is natural to set $C_i$ as the *observer's frame*, i.e.,

$$\boldsymbol{F}^{C_i} = \begin{bmatrix} \boldsymbol{f}^{C_i} \\ \boldsymbol{\tau}^{C_i} \end{bmatrix}$$

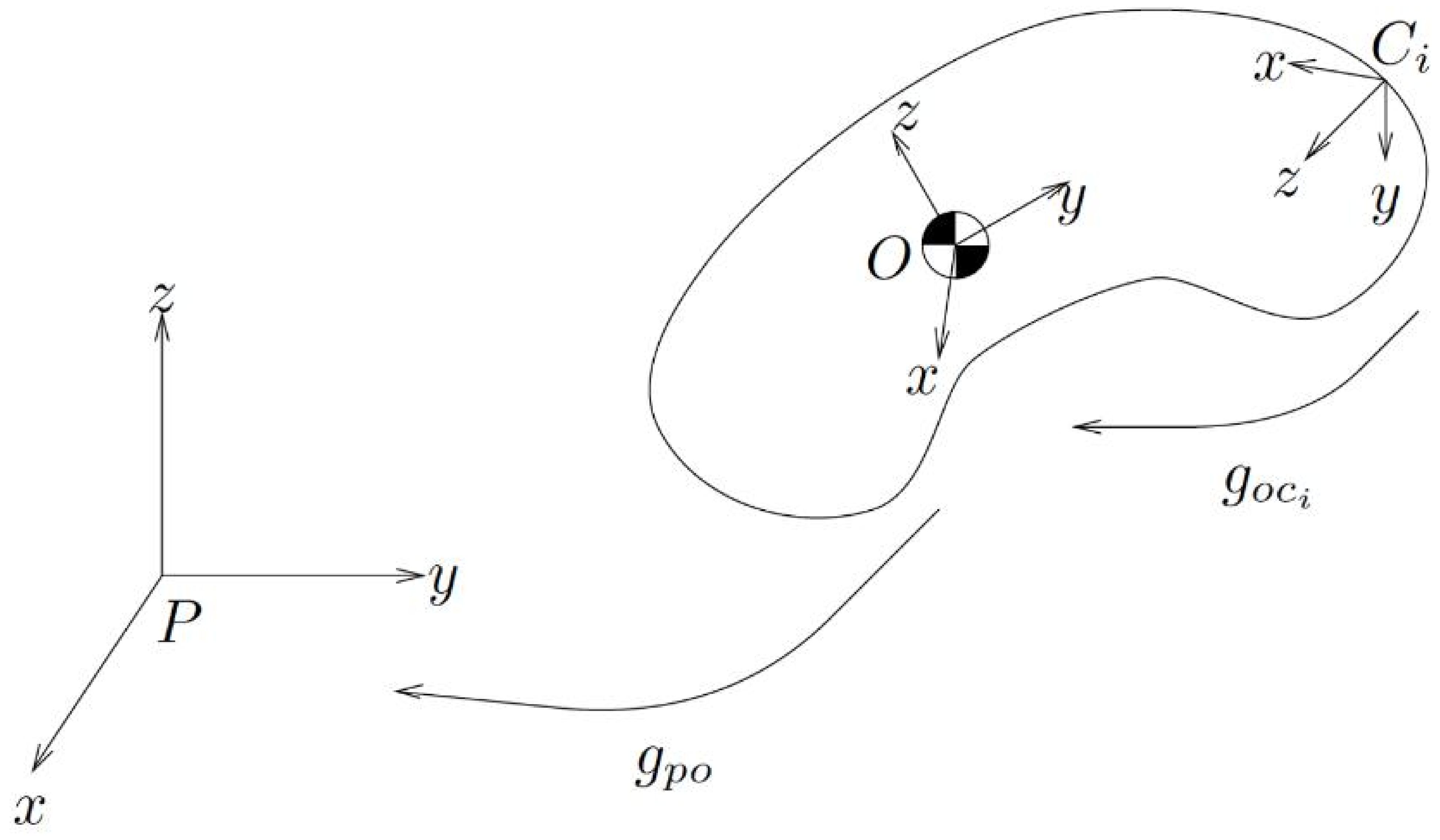


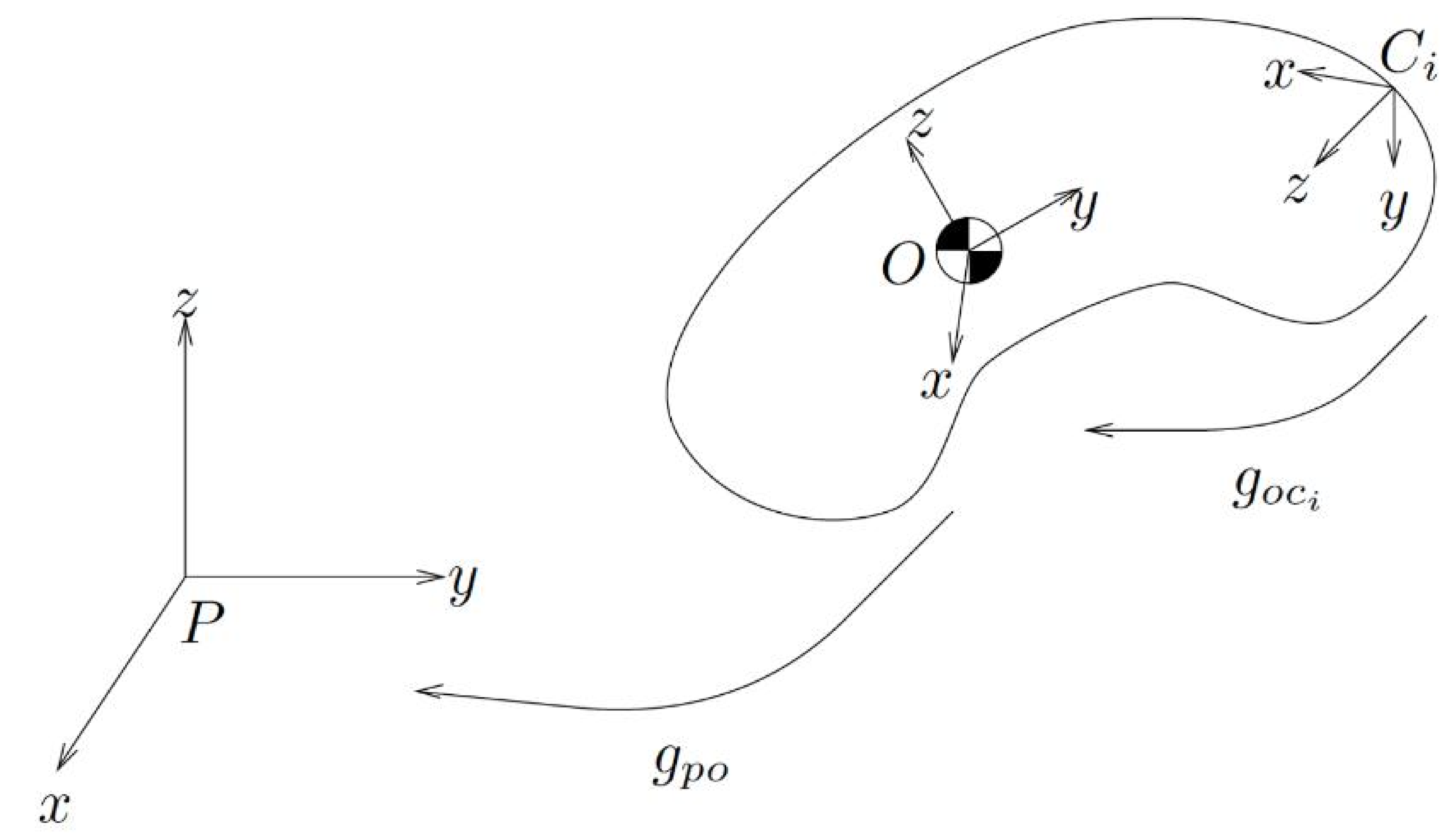

Figure 5.2: Coordinate frames for contact and object forces.

# Some Kinds of Contact Forces

| Contact Type | Frictionless point contact | Point contact with friction | Soft-finger |
|---|---|---|---|
| $\boldsymbol{F}^C$ | $\begin{bmatrix} 0 \\ 0 \\ f_z \\ 0 \\ 0 \\ 0 \end{bmatrix}$ | $\begin{bmatrix} f_x \\ f_y \\ f_z \\ 0 \\ 0 \\ 0 \end{bmatrix}$ | $\begin{bmatrix} f_x \\ f_y \\ f_z \\ 0 \\ 0 \\ \tau_z \end{bmatrix}$ |

# Adding Forces and Torques

- Suppose we have calculated $\boldsymbol{F}^{C_i}$ at each contact (will learn later)
- What is the combined force and torque?

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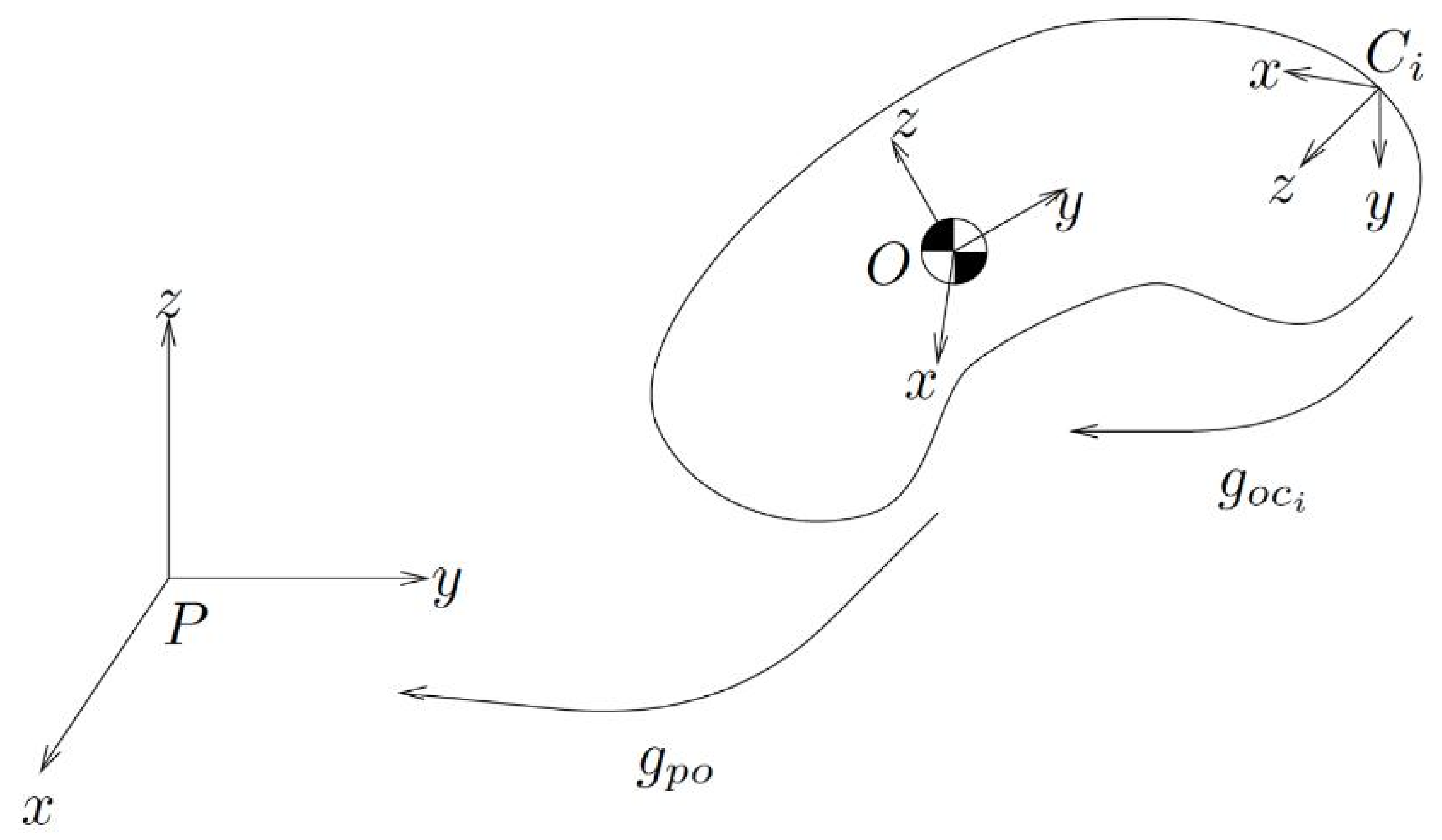


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# Adding Forces and Torques

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- What is the combined force and torque?
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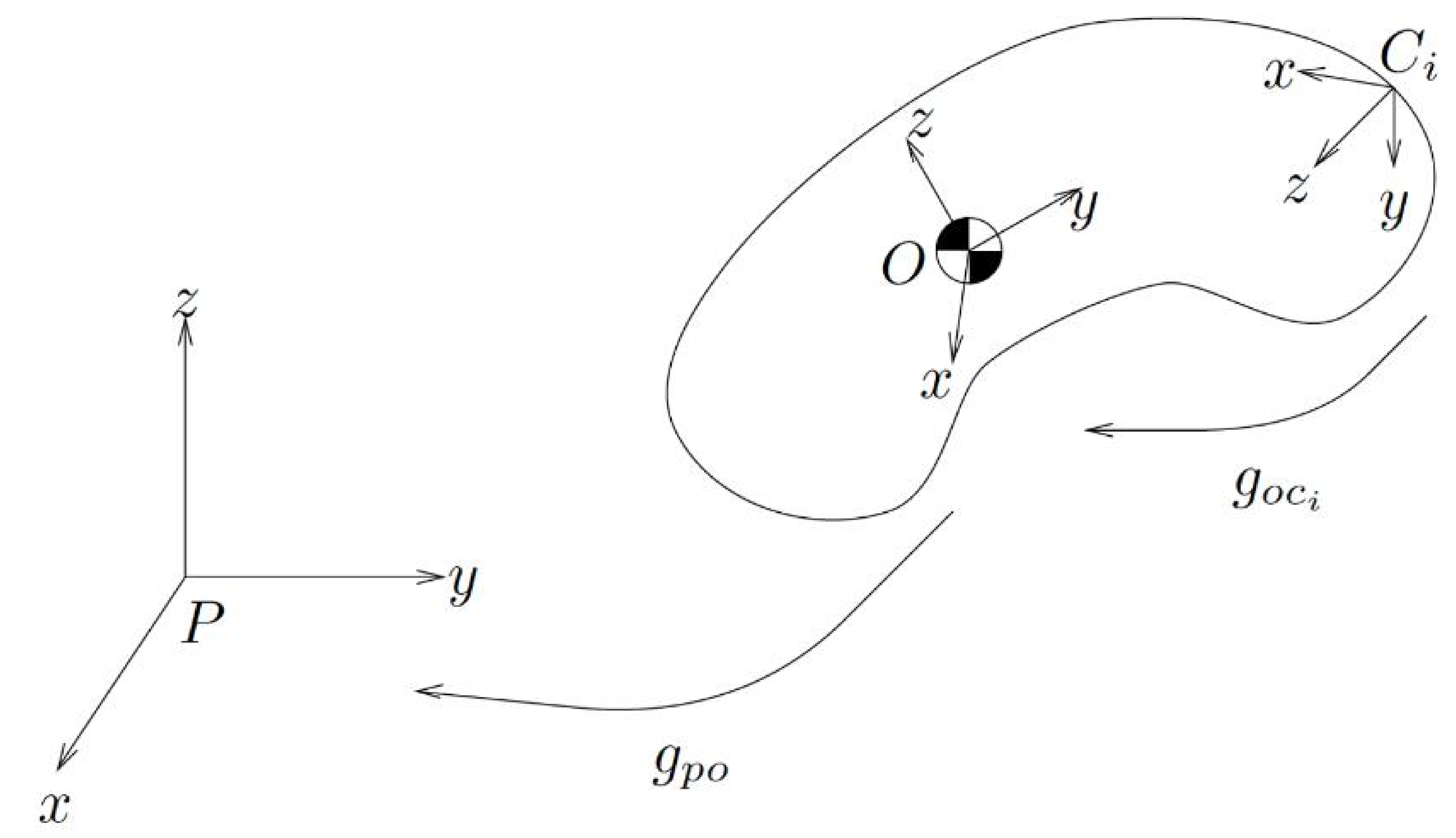
- However, we can change all to the same frame (e.g., body frame) and add together!

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# Change of Observer's Frame for Force and Torque

Consider the question of changing the observer's frame for force and torque

- We would relate
  - $\boldsymbol{f}^1$ and $\boldsymbol{f}^2$
  - $\boldsymbol{\tau}^1 = \boldsymbol{r}^1 \times \boldsymbol{f}^1$ and $\boldsymbol{\tau}^2 = \boldsymbol{r}^2 \times \boldsymbol{f}^2$
- Note that

$$\begin{aligned} \boldsymbol{r}^2 &= R_{2\to1}\boldsymbol{r}^1 + \boldsymbol{t}_{2\to1} \\ \boldsymbol{f}^2 &= R_{2\to1}\boldsymbol{f}^1 \end{aligned}$$

- Plug in the definition, and we derive that

$$\begin{bmatrix} \boldsymbol{f}^2 \\ \boldsymbol{\tau}^2 \end{bmatrix} = \begin{bmatrix} R_{2\to1} & 0 \\ [\boldsymbol{t}_{2\to1}]R_{2\to1} & R_{2\to1} \end{bmatrix} \begin{bmatrix} \boldsymbol{f}^1 \\ \boldsymbol{\tau}^1 \end{bmatrix} = (\mathrm{Ad}_{1\to2})^{\mathrm{T}} \begin{bmatrix} \boldsymbol{f}^1 \\ \boldsymbol{\tau}^1 \end{bmatrix}$$

# Change of Observer's Frame for Force and Torque

- Define $\boldsymbol{F}^o = \begin{bmatrix} \boldsymbol{f}^o \\ \boldsymbol{\tau}^o \end{bmatrix}$, then formula for change of frame is:

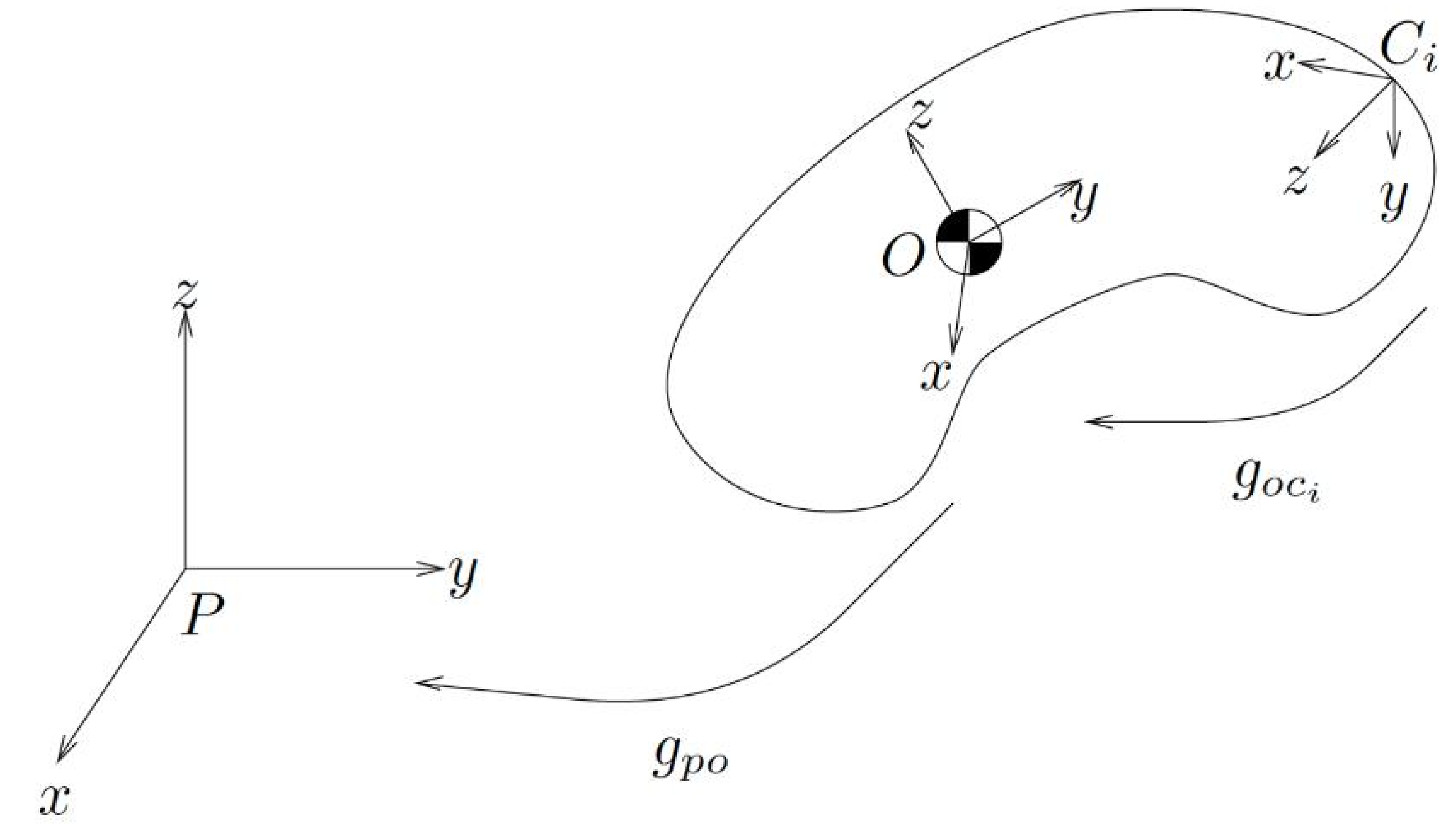
$$\boldsymbol{F}^2 = (\mathrm{Ad}_{1\to 2})^{\mathrm{T}} \boldsymbol{F}^1 \qquad \text{(change of frame)}$$

- Using definitions and frame change equations, it is easy to verify that the following equation to compute the **power** of the system input (change rate of kinetic energy):

$$(\boldsymbol{F}^b)^T \boldsymbol{\xi}^b = (\boldsymbol{F}^o)^T \boldsymbol{\xi}^o = \frac{\mathrm{d}T}{\mathrm{d}t} \qquad \text{(system input power)}$$

# Solution to Adding Forces and Torques

$$\boldsymbol{F}^b = \sum_{i=1}^{k} [\mathrm{Ad}_{\mathrm{C_i}\to\mathrm{b}}]^{\mathrm{T}} \boldsymbol{F}^{\mathrm{C_i}}$$

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# Change of Observer's Frame for Momentum and Angular Momentum

Consider the question of changing the observer's frame for momentum and angular momentum

- We would relate
  - $\boldsymbol{p}^1 = m\boldsymbol{v}^1$ and $\boldsymbol{p}^2 = m\boldsymbol{v}^2$
  - $\boldsymbol{L}^1 = \boldsymbol{r}^1 \times m\boldsymbol{v}^1$ and $\boldsymbol{L}^2 = \boldsymbol{r}^2 \times m\boldsymbol{v}^2$
- Note that

$$\boldsymbol{r}^2 = R_{2\to1}\boldsymbol{r}^1 + \boldsymbol{t}_{2\to1}$$
$$\boldsymbol{v}^2 = R_{2\to1}\boldsymbol{v}^1$$

- The same derivation as force and torque pair, and we get $\begin{bmatrix}\boldsymbol{p}^2\\ \boldsymbol{L}^2\end{bmatrix} = (\mathrm{Ad}_{1\to2})^{\mathrm{T}}\begin{bmatrix}\boldsymbol{p}^1\\ \boldsymbol{L}^1\end{bmatrix}$

*Read by Yourself*

# Change of Observer's Frame for Momentum and Angular Momentum

- Define $\boldsymbol{P}^o = \begin{bmatrix} \boldsymbol{p}^o \\ \boldsymbol{L}^o \end{bmatrix}$, and the formula for change of frame is:

$$\boldsymbol{P}^2 = (\mathrm{Ad}_{1\to 2})^{\mathrm{T}} \boldsymbol{P}^1 \qquad \text{(change of frame)}$$

- Note: similar to linear momentum that $\boldsymbol{p}^o = \frac{\mathrm{d}T}{\mathrm{d}\boldsymbol{v}^o}$ for translation-only motion, it is straight-forward to verify that

$$\boldsymbol{P}^o = \frac{dT}{d\boldsymbol{\xi}^o} = \mathfrak{M}^o \xi^o \qquad \text{(generalized angular momentum)}$$

# Summary

- We have learned basic concepts for body motion dynamics
  - Properties of objects: mass, rotational inertia
  - Motion state: momentum, angular momentum
  - Action: force, torque
  - Energy perspective: kinetic energy
- We have also introduced various equations for changing the observer's frame

# Forward and Inverse Dynamics

# Dynamics Example: Grasp

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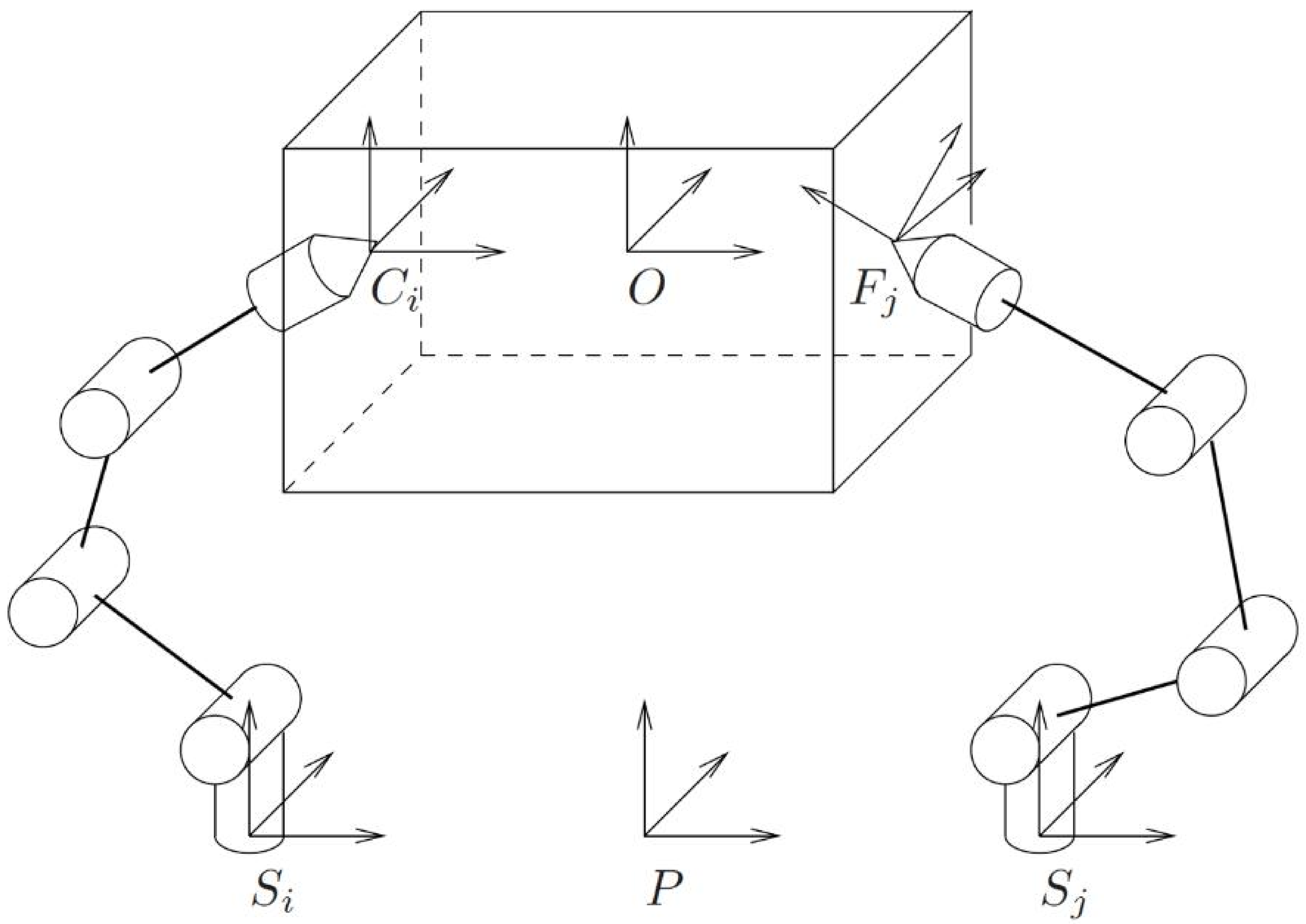


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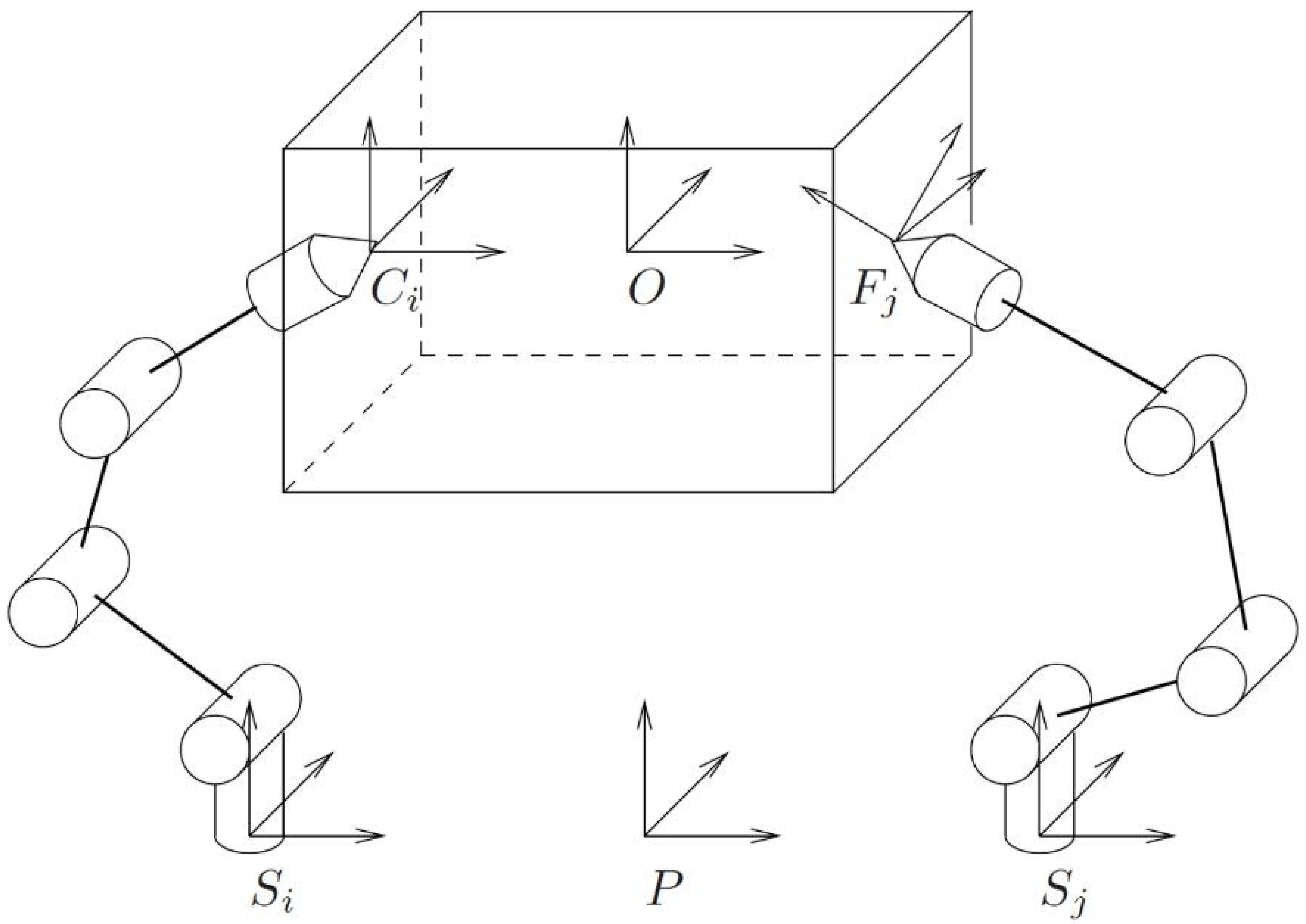


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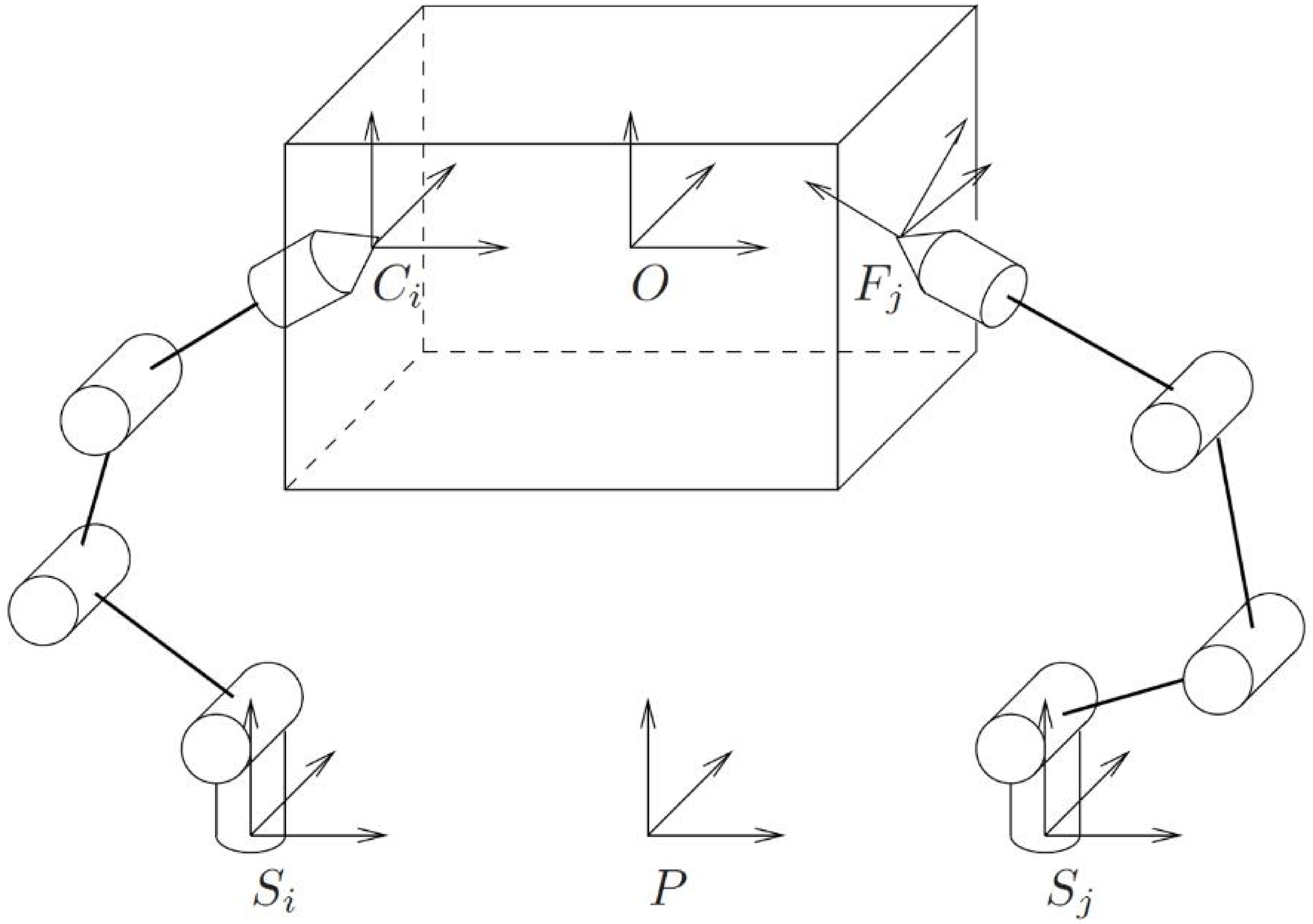


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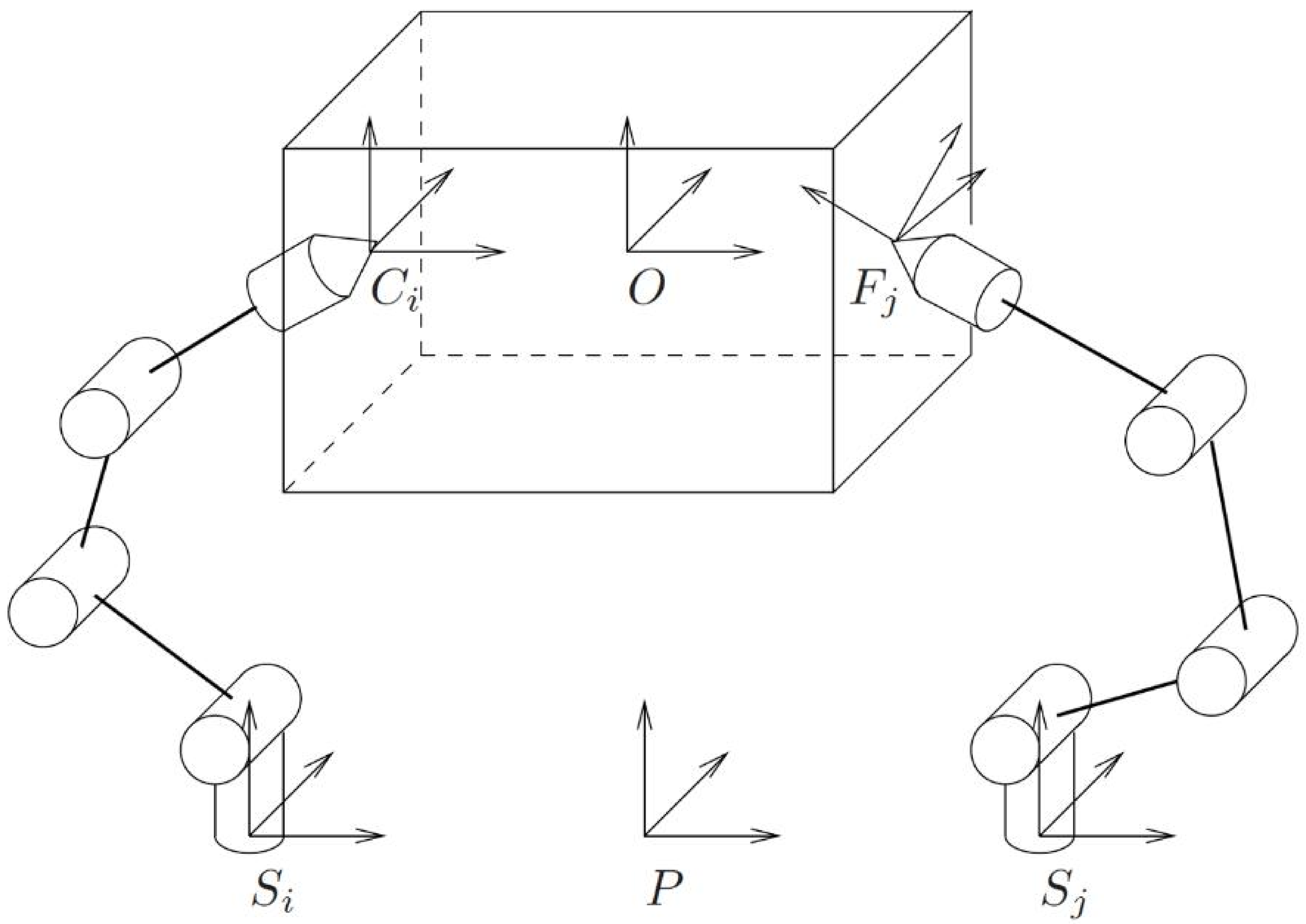


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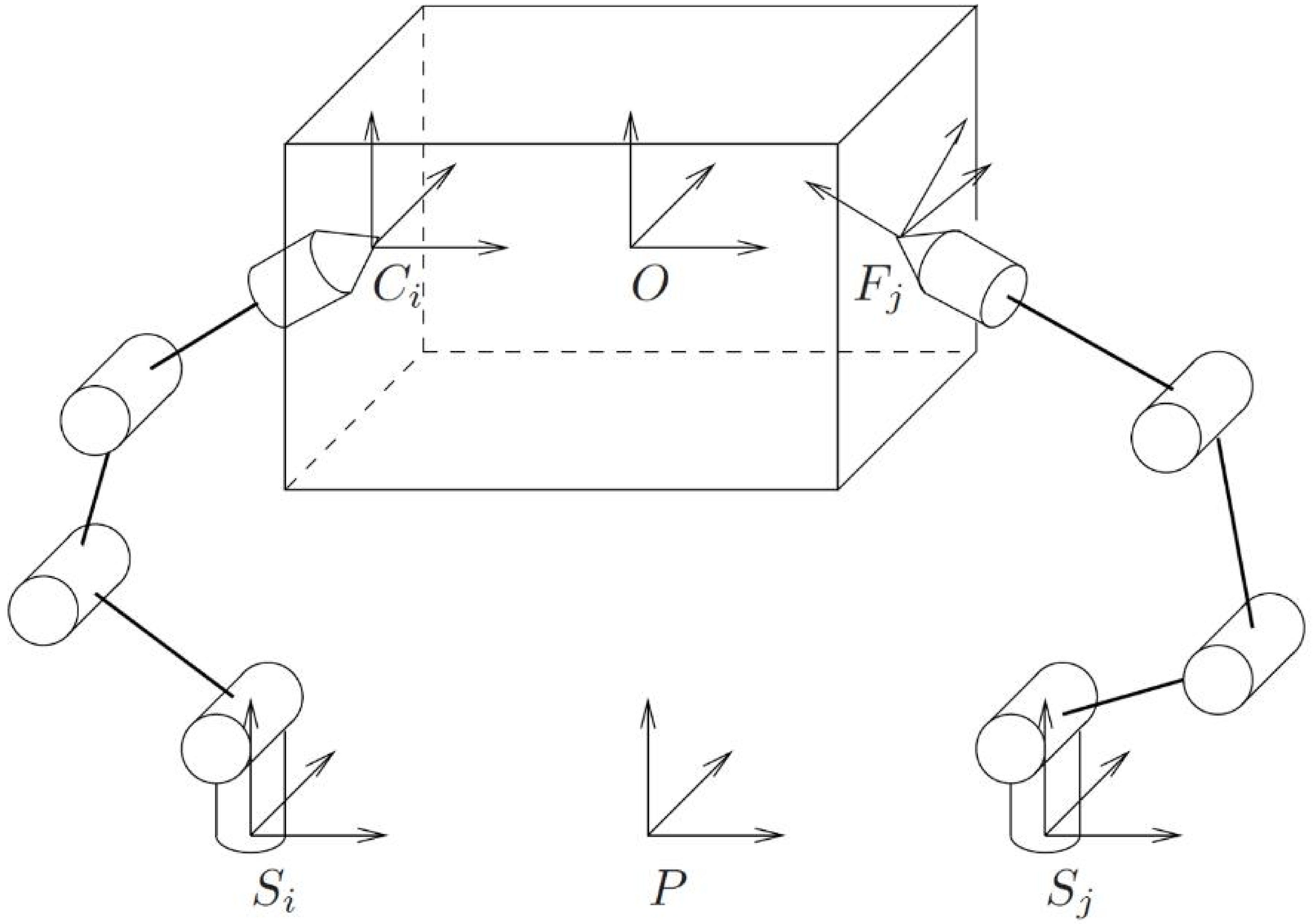


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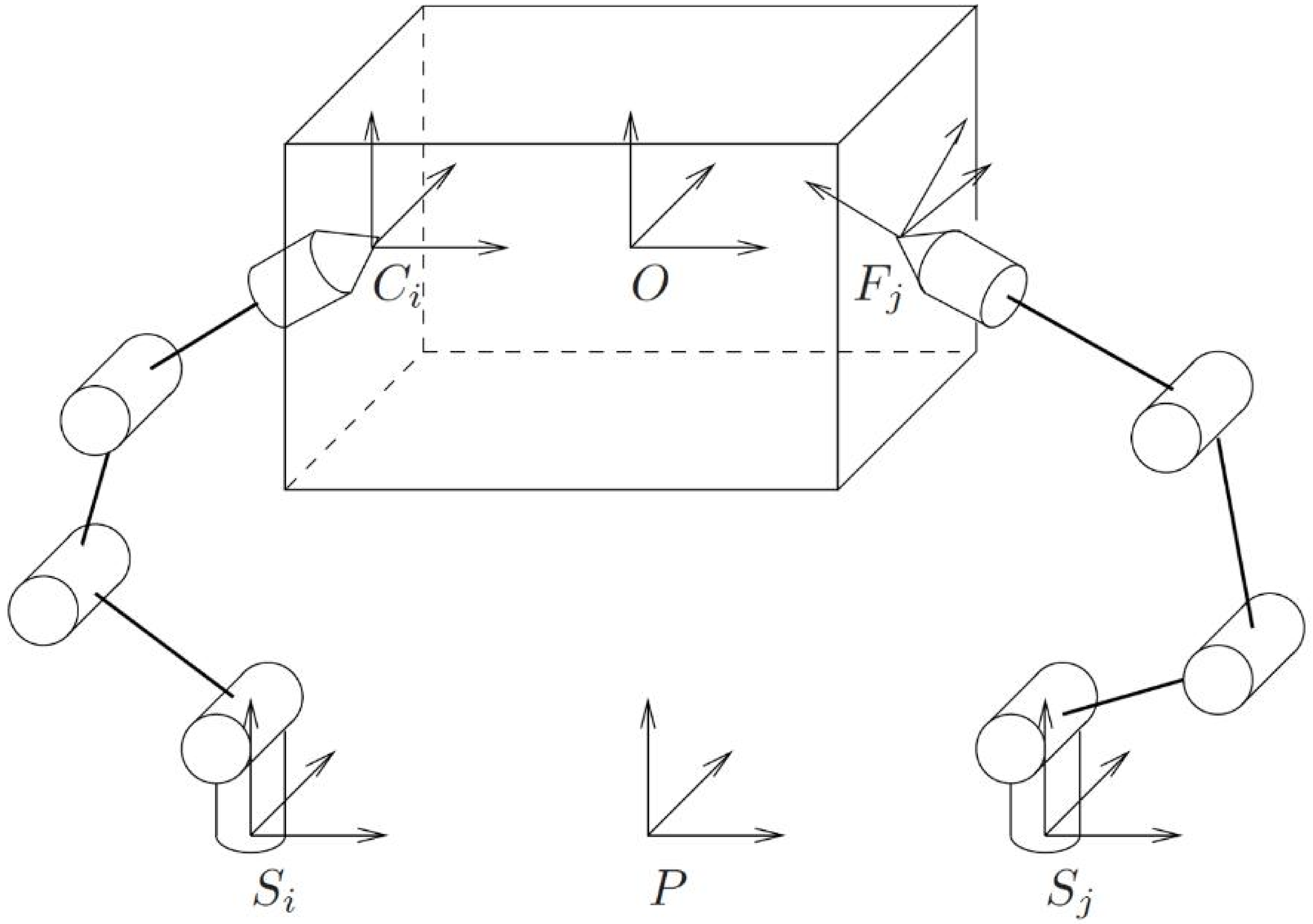
**Q1: How to compute force at the tips from the torques at joints?**

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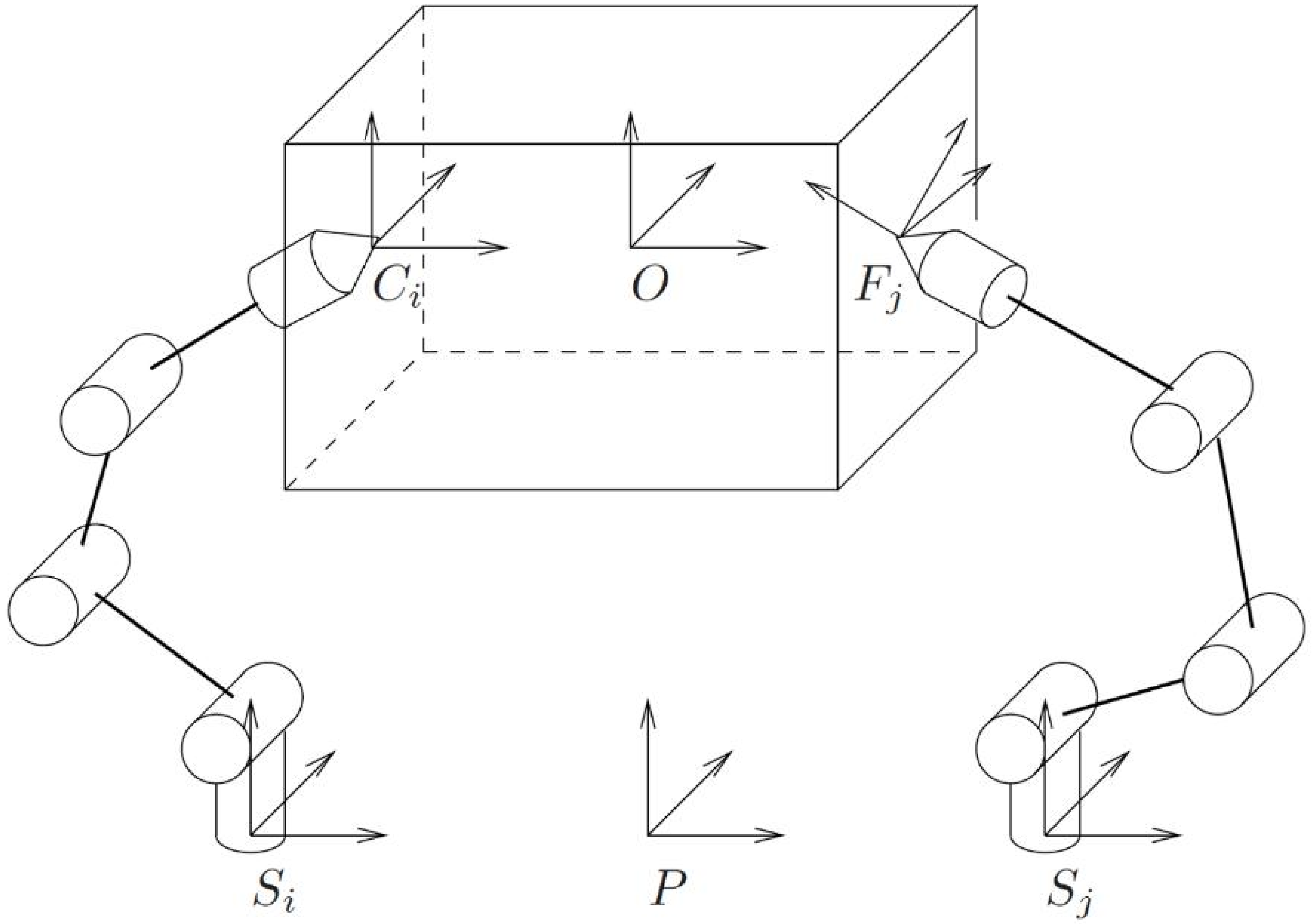
**Q2: To keep the box static, what is the balance condition?**

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- Task
  - **Forward dynamics:** Determine acceleration $\ddot{\theta}$ given the state $(\theta, \dot{\theta})$ and the joint forces/torques

$$\ddot{\theta} = f(\tau; \theta, \dot{\theta})$$

  - **Inverse dynamics:** Finding torques/forces given state $\theta, \dot{\theta}$ and desired acceleration $\ddot{\theta}$

$$\tau = g(\ddot{\theta}; \theta, \dot{\theta})$$

# Lagrangian vs. Newton-Euler Methods

- There are typically two ways to derive the equation of motion for an open-chain robot: Lagrangian method and Newton-Euler method

### Lagrangian Formulation

- Energy-based method
- Dynamic equations in closed form
- Often used for study of dynamic properties and analysis of control methods

### Newton-Euler Formulation

- Balance of forces/torques
- Dynamic equations numeric/recursive form
- Often used for numerical solution of forward/inverse dynamics