L7-2: Basic Concepts of Rigid-Body Dynamics

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Agenda

- Angular Momentum and Rotational Inertia
- Torque

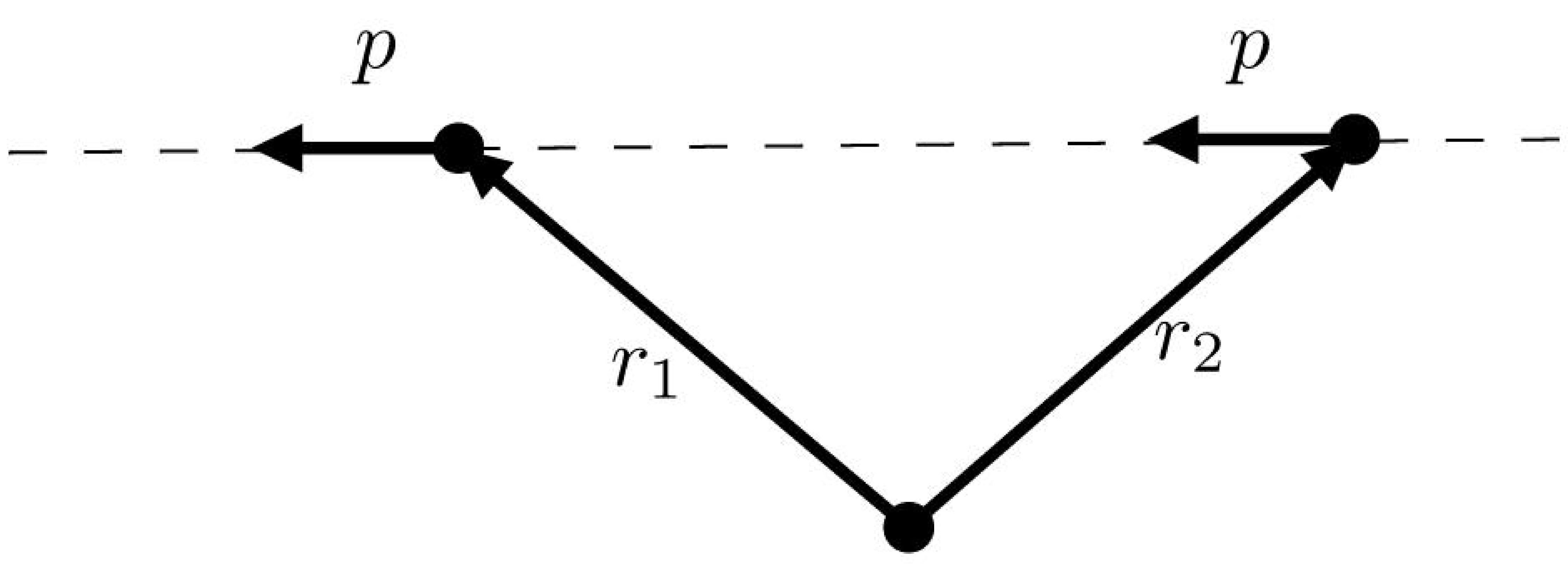
click to jump to the section.

Angular Momentum and Rotational Inertia

Angular Momentum of Point Mass

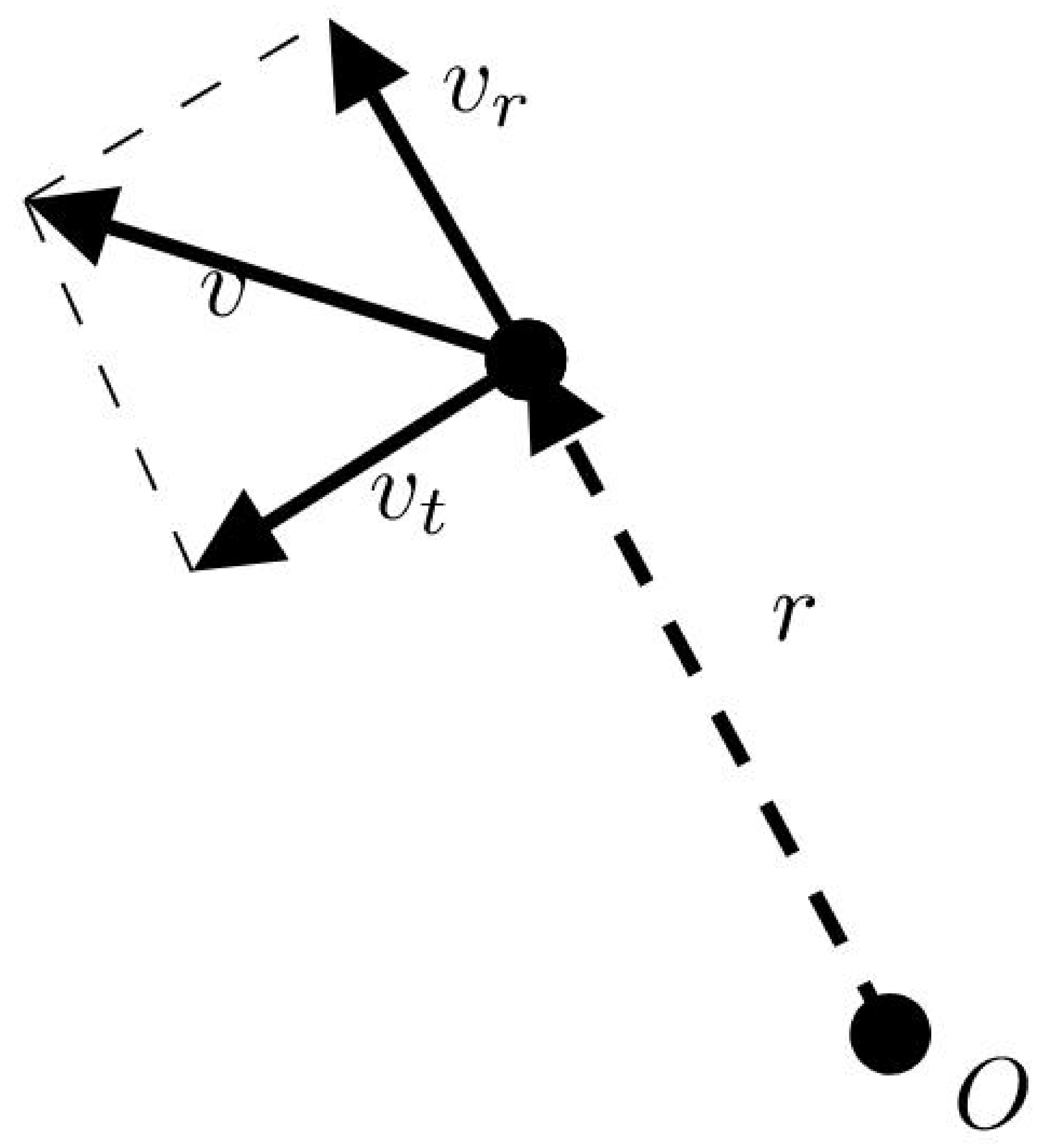
- Assume a point mass m that has a momentum $\boldsymbol{p}^o$
- Assume a vector from *the origin of the observer's frame* O to the point mass $\boldsymbol{r}^o$
- Angular momentum:

$$\boldsymbol{L}^o = \boldsymbol{r}^o \times \boldsymbol{p}^o$$



Rotational Inertia Preparation

$\boldsymbol{v}$ can be decomposed into tangential velocity $\boldsymbol{v}_t$ and radial velocity $\boldsymbol{v}_r$



$$\boldsymbol{r} \times \boldsymbol{v} = \boldsymbol{r} \times (\boldsymbol{v}_t + \boldsymbol{v}_r) = \boldsymbol{r} \times \boldsymbol{v}_t = \boldsymbol{r} \times (\boldsymbol{\omega} \times \boldsymbol{r})$$

Rotational Inertia of Point Mass

$$
\begin{aligned}
\boldsymbol{L}^o &= \boldsymbol{r}^o \times \boldsymbol{p}^o = \boldsymbol{r}^o \times (m\boldsymbol{v}^o) = m\boldsymbol{r}^o \times (\boldsymbol{\omega}^o \times \boldsymbol{r}^o) \\
&= -m\boldsymbol{r}^o \times (\boldsymbol{r}^o \times \boldsymbol{\omega}^o) = -m[\boldsymbol{r}^o][\boldsymbol{r}^o]\boldsymbol{\omega}^o
\end{aligned}
$$

Angular momentum depends on the choice of the observer's frame!

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Angular momentum depends on the choice of the observer's frame!

- Recall that a momentum, such as $\boldsymbol{p}$, is a product of inertia and velocity
- We define the rotational inertia similarly. The rotation inertia for a point mass is

$$\boldsymbol{I}^o = -m[\boldsymbol{r}^o][\boldsymbol{r}^o] = \begin{bmatrix} m(r_y^2 + r_z^2) & -mr_xr_y & -mr_xr_z \\ -mr_xr_y & m(r_x^2 + r_z^2) & -mr_yr_z \\ -mr_xr_z & -mr_yr_z & m(r_x^2 + r_y^2) \end{bmatrix} \in \mathbb{R}^{3\times 3}$$

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- Then,

$$\boldsymbol{L}^o = \boldsymbol{I}^o\boldsymbol{\omega}^o$$

Angular Momentum and Inertia of Rigid Body

- Let us view rigid body as *a system of particles* whose relative positions are fixed (no deformation).
- Define the angular momentum of a body by aggregating from volume elements:

$$\boldsymbol{L}^o = \int_{x^o \in B} \mathrm{d}\{\boldsymbol{r}^o(x) \times \boldsymbol{p}^o(x^o)\} = \int_{x^o \in B} \mathrm{d}\{\boldsymbol{r}^o(x) \times m(x^o)\boldsymbol{v}^o(x^o)\}$$

- One more step:

$$\boldsymbol{L}^o = \int_{x^o \in B} -\mathrm{d}\{m^o(x^o)[\boldsymbol{r}^o(x^o)][\boldsymbol{r}^o(x^o)]\boldsymbol{\omega}^o\} = \left(\int_{x^o \in B} -\mathrm{d}\{m(x^o)[\boldsymbol{r}^o(x^o)][\boldsymbol{r}^o(x^o)]\}\right)\boldsymbol{\omega}^o$$

Angular Momentum and Inertia of Rigid Body

- Particularly, if we choose the origin of the observer's frame O at the *center of mass*:

$$\boldsymbol{L}^b = \boldsymbol{I}^b \boldsymbol{\omega}^b \qquad \text{(body angular momentum)}$$

where

$$\boldsymbol{I}^b = \int_{x^b \in B} -\mathrm{d}V \{\rho(x^b)[\boldsymbol{r}^b(x^b)][\boldsymbol{r}^b(x^b)]\} \qquad \text{(body inertia)}$$

and center of mass

$$x^o_{cm} = \frac{\int \boldsymbol{r}^o \rho d\mathbf{V}}{\int \rho d\mathbf{V}} \qquad \text{(center of mass)}$$

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- Since $\mathcal{F}_{b(t)}$ is tightly binded to the body, $\boldsymbol{I}^b$ does not change w.r.t. time and is a basic property of the object.

Computation of Rigid Body Inertia

$$
\begin{aligned}
\boldsymbol{I}^b &= \int_{x^b \in B} -\mathrm{d}V \rho(\boldsymbol{x}^b)[\boldsymbol{r}^b(\boldsymbol{x}^b)][\boldsymbol{r}^b(\boldsymbol{x}^b)] \\
&= \begin{bmatrix} \int \rho(r_y^2 + r_z^2) d\boldsymbol{V} & -\int \rho r_x r_y d\boldsymbol{V} & -\int \rho r_x r_z d\boldsymbol{V} \\ -\int \rho r_x r_y d\boldsymbol{V} & \int \rho(r_x^2 + r_z^2) d\boldsymbol{V} & -\int \rho r_y r_z d\boldsymbol{V} \\ -\int \rho r_x r_z d\boldsymbol{V} & -\int \rho r_y r_z d\boldsymbol{V} & \int \rho(r_y^2 + r_x^2) d\boldsymbol{V} \end{bmatrix}
\end{aligned}
$$

- Given uniform density, the integral can be computed analytically for watertight meshes

Fast Inertia Computation

- Divergence theorem! Let $\boldsymbol{F}:\mathbb{R}^3\rightarrow\mathbb{R}^3$, $\int_V \nabla\cdot\boldsymbol{F}dV=\oint_S \boldsymbol{F}\cdot\boldsymbol{n}dS$
- An example: a term of $\boldsymbol{I}$, which is $-\rho\int_{\boldsymbol{V}} \boldsymbol{r}_y\boldsymbol{r}_z d\boldsymbol{V}$
  Let $\boldsymbol{F}(\boldsymbol{r}_x,\boldsymbol{r}_y,\boldsymbol{r}_z)=\begin{bmatrix}\boldsymbol{r}_x\boldsymbol{r}_y\boldsymbol{r}_z & 0 & 0\end{bmatrix}^T$

$$\nabla\cdot\boldsymbol{F}=\boldsymbol{r}_y\boldsymbol{r}_z$$

The integral becomes

$$\oint_S \boldsymbol{F}\cdot\boldsymbol{n}dS$$

Now we only need to do 2D integral over triangles.

Read by yourself

Mass Properties

- Observe $\boldsymbol{I}^b = \int_{\boldsymbol{r}^b \in B} -\mathrm{d}\boldsymbol{V} \rho(\boldsymbol{r}^b)[\boldsymbol{r}^b][\boldsymbol{r}^b]$
- Although the origin is always at the center of mass, if we change the orientation of body frame axes, $\boldsymbol{I}^b$ may change!
- How will it change, then?

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- How will it change, then?
- If we rotate the frame by R^T and obtain a new frame b', then

$$\boldsymbol{I}^{b'} = \int_{\boldsymbol{r}^b \in B} -\mathrm{d}\boldsymbol{V} \rho(\boldsymbol{r}^b)[R\boldsymbol{r}^b][R\boldsymbol{r}^b] = \int_{\boldsymbol{r}^b \in B} -\mathrm{d}\boldsymbol{V} \rho(\boldsymbol{r}^b) R[\boldsymbol{r}^b][\boldsymbol{r}^b]R^T = R\boldsymbol{I}^b R^T$$

where the second equality follows $[Rr] = R[r]R^T$ for $R \in \mathbb{SO}^3$. Again, **similarity transformation**!

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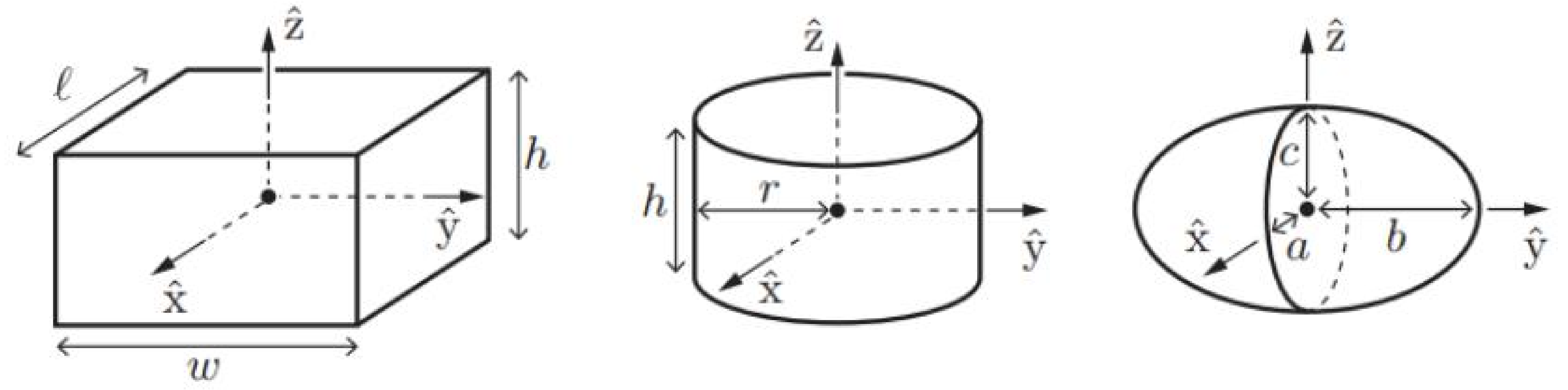
Conclusion: Rigid-transformation does not change the eigen properties of $\boldsymbol{I}^b$

Mass Properties

- $\boldsymbol{I}^b$ admits eigen-decomposition
  - The eigenvectors are called **principal axes**.
  - The eigenvalues (I_1, I_2, I_3) are called the **principal moments of inertia**.



rectangular parallelepiped:	circular cylinder:	ellipsoid:
volume $= abc$,	volume $= \pi r^2 h$,	volume $= 4\pi abc/3$,
$\mathcal{I}_{xx} = \mathfrak{m}(w^2+h^2)/12$,	$\mathcal{I}_{xx} = \mathfrak{m}(3r^2+h^2)/12$,	$\mathcal{I}_{xx} = \mathfrak{m}(b^2+c^2)/5$,
$\mathcal{I}_{yy} = \mathfrak{m}(\ell^2+h^2)/12$,	$\mathcal{I}_{yy} = \mathfrak{m}(3r^2+h^2)/12$,	$\mathcal{I}_{yy} = \mathfrak{m}(a^2+c^2)/5$,
$\mathcal{I}_{zz} = \mathfrak{m}(\ell^2+w^2)/12$	$\mathcal{I}_{zz} = \mathfrak{m}r^2/2$	$\mathcal{I}_{zz} = \mathfrak{m}(a^2+b^2)/5$

(from: https://www.cnblogs.com/21207-iHome/p/7765508.html)

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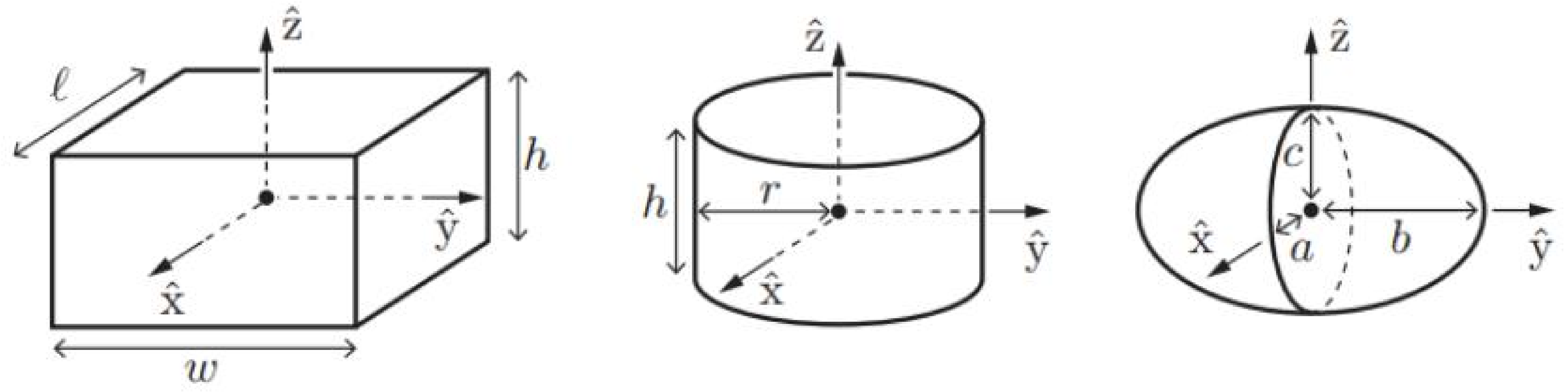
- x_{cm} and principal axes form a **body frame** that is intrinsic to the object



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ellipsoid:
volume $= 4\pi abc/3$,
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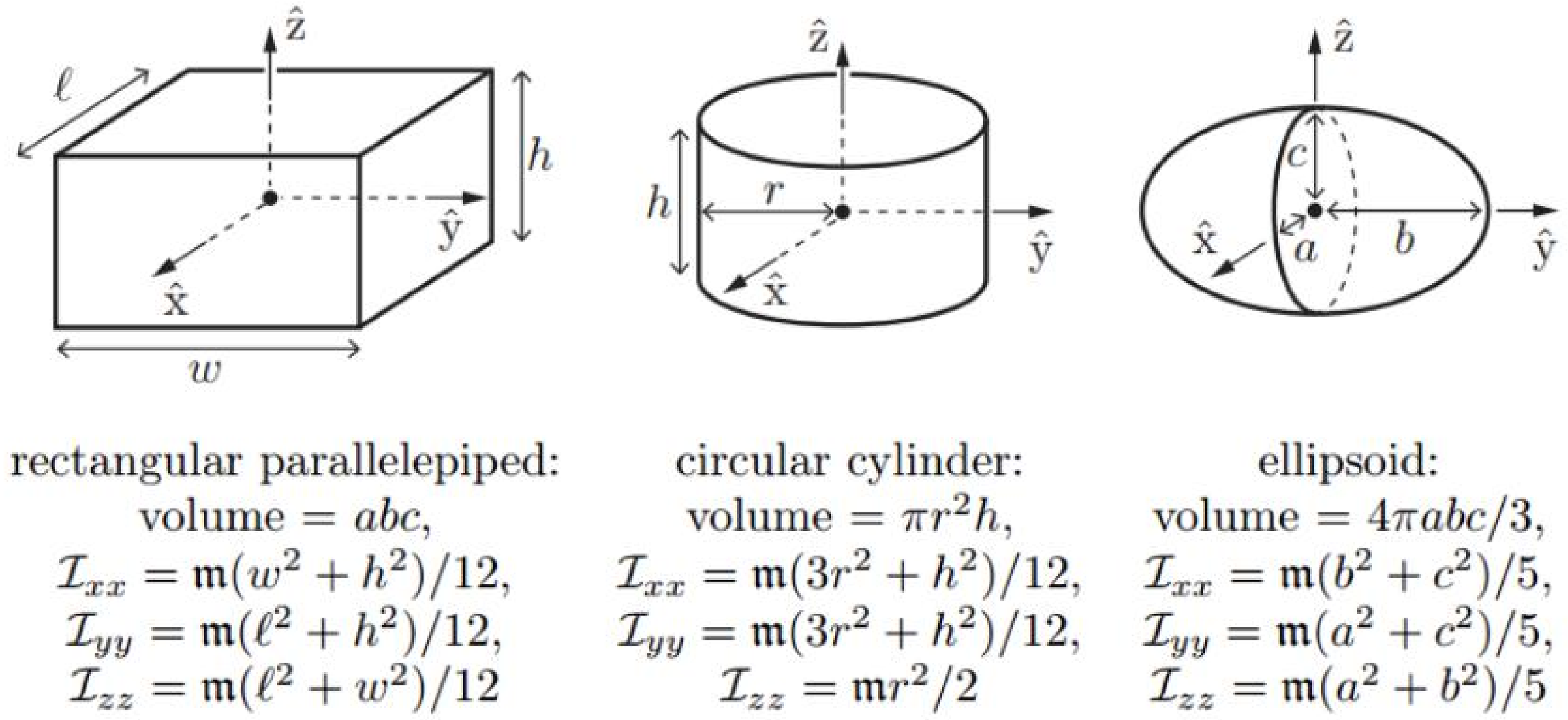
- x_{cm} and principal axes form a **body frame** that is intrinsic to the object
- **x_{cm}, principal axes, m, I_1, I_2, I_3 fully determine the behavior of a rigid body under external forces**



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Quiz

Suppose an object is moving in space (rotating and translating), which of the following quantities may change during the motion. (Assume all quantities are measured w.r.t. a static spatial frame)

- A. principal axes (observed from the spatial frame)
- B. x_{cm} (observed from the spatial frame)
- C. m
- D. I_1, I_2, I_3

Torque

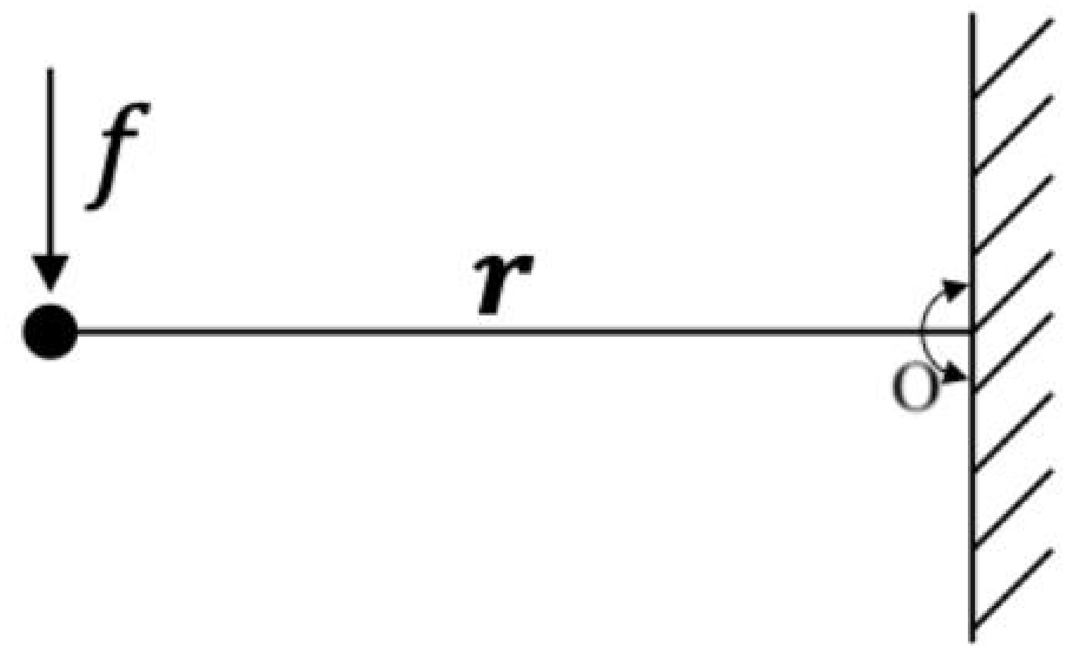
Torque

- Consider a simple example on the right.
- Recall how we define the angular momentum $\boldsymbol{L}^o$ for *point mass*:

$$\boldsymbol{L}^o = \boldsymbol{r}^o \times \boldsymbol{p}^o = \boldsymbol{r}^o \times (m\boldsymbol{v}^o) \tag{1}$$

- We have also derived that

$$\boldsymbol{L}^o = \boldsymbol{I}^o \boldsymbol{\omega}^o \tag{2}$$



Example: a point mass is fixed at the end of a light stick mounted on the wall. At the moment of analysis, it has velocity $\boldsymbol{v}$.

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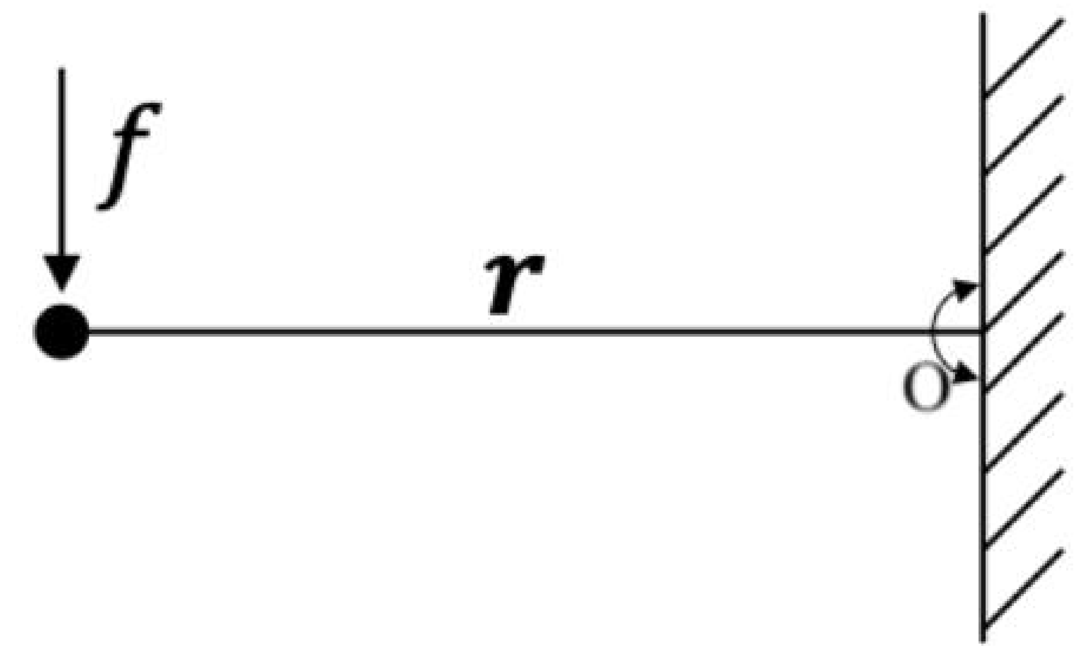
- We use the time derivative of $\boldsymbol{L}^o$ to define **torque**, denoted by τ^o
  1. By (1), $\tau^o = \dot{\boldsymbol{L}}^o = \dot{\boldsymbol{r}}^o \times (m\boldsymbol{v}^o) + \boldsymbol{r}^o \times \boldsymbol{f}^o = \boldsymbol{r}^o \times \boldsymbol{f}^o$, because $\dot{\boldsymbol{r}}^o \parallel \boldsymbol{v}^o$
  2. By (2), $\tau = \frac{\mathrm{d}(\boldsymbol{I}^o\boldsymbol{\omega}^o)}{\mathrm{d}t}$



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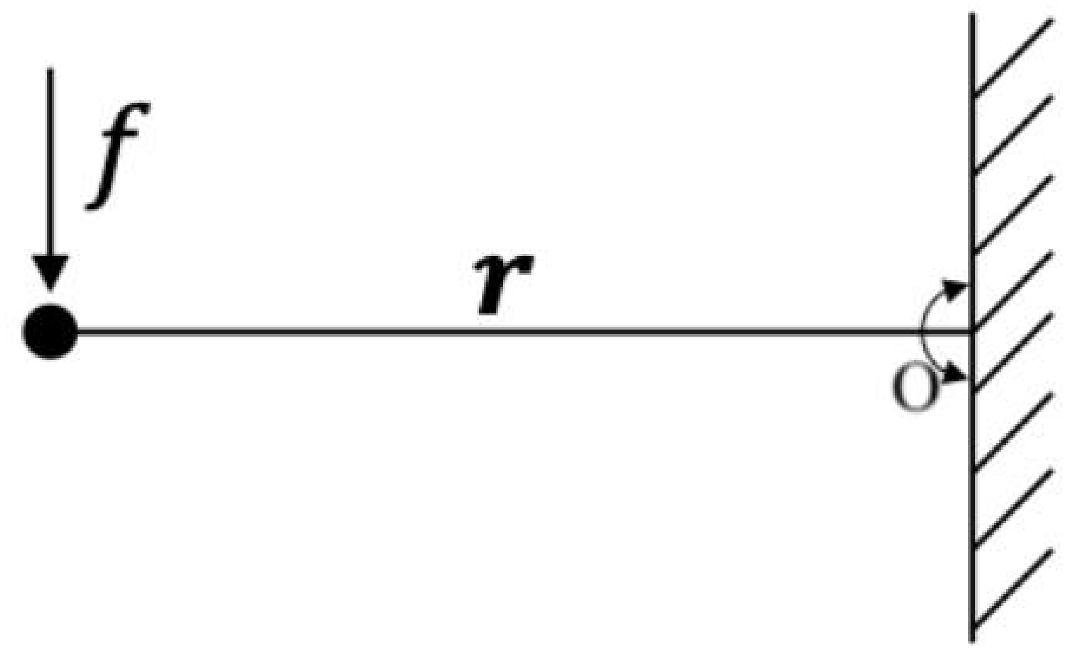
  2. By (2), $\boldsymbol{\tau} = \frac{\mathrm{d}(\boldsymbol{I}^o \boldsymbol{\omega}^o)}{\mathrm{d}t}$
- Torque describes how fast the angular momentum changes (from 2). Torque also relates the change with the cause: an external power input (from 1).



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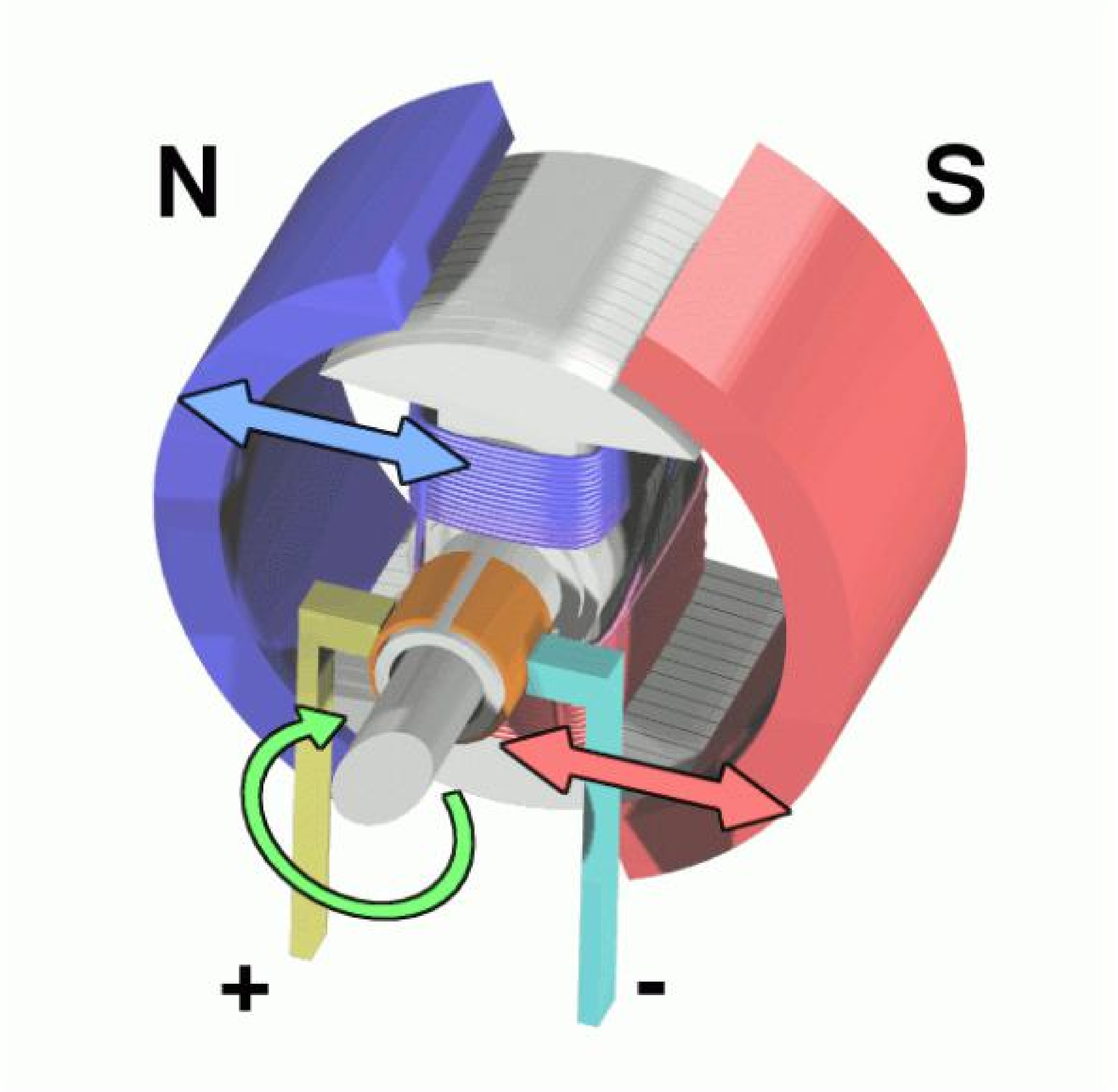
Torque

- In the example of point mass, we showed the equality of two torque computations
  - the change rate of $\boldsymbol{L}$
  - the input to the system
- For general rigid-body systems, the equality is also true
- For robotic manipulation, torque is the most common description of system input



https://en.wikipedia.org/wiki/Electric_motor

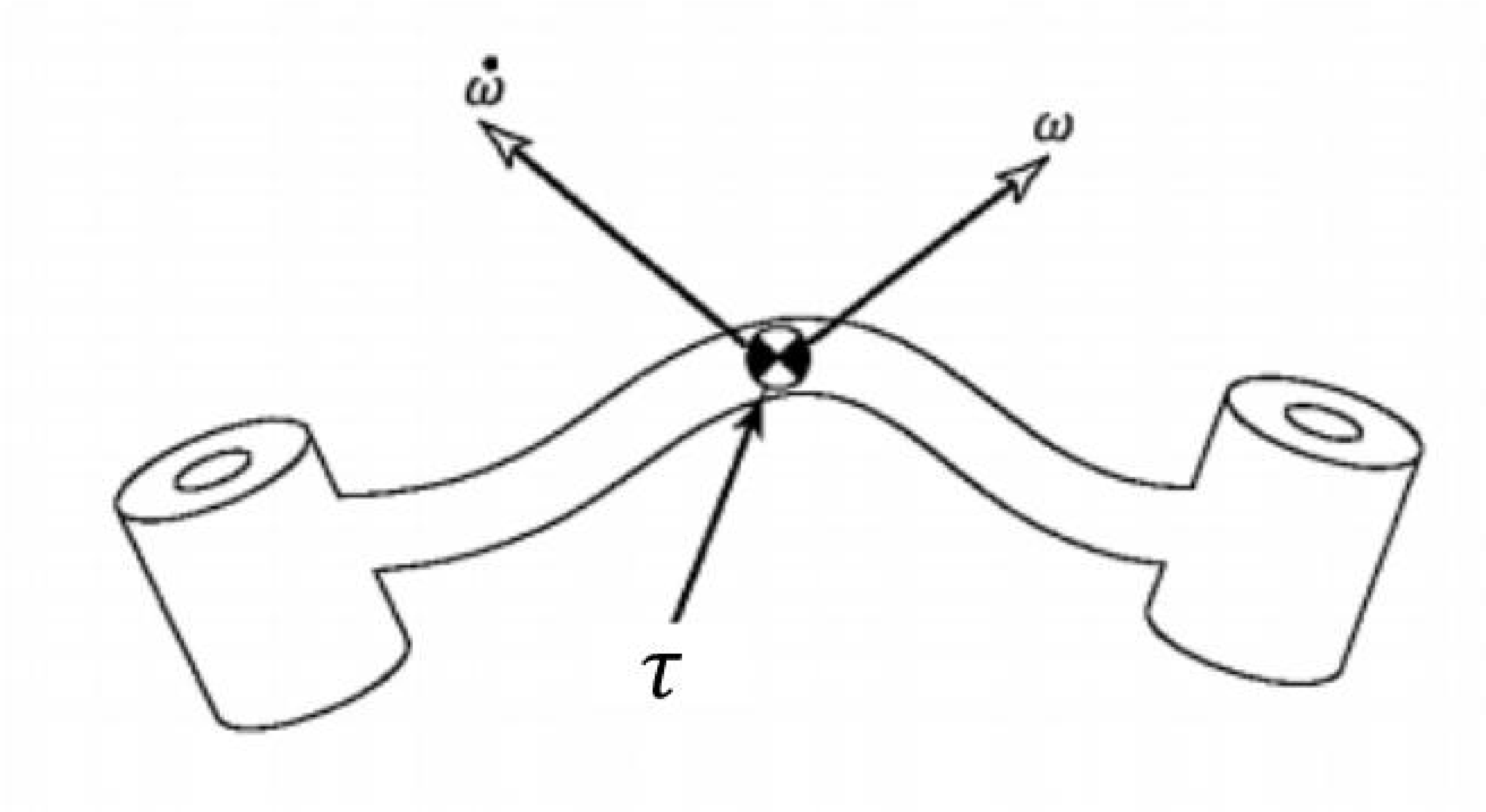
Euler Equation

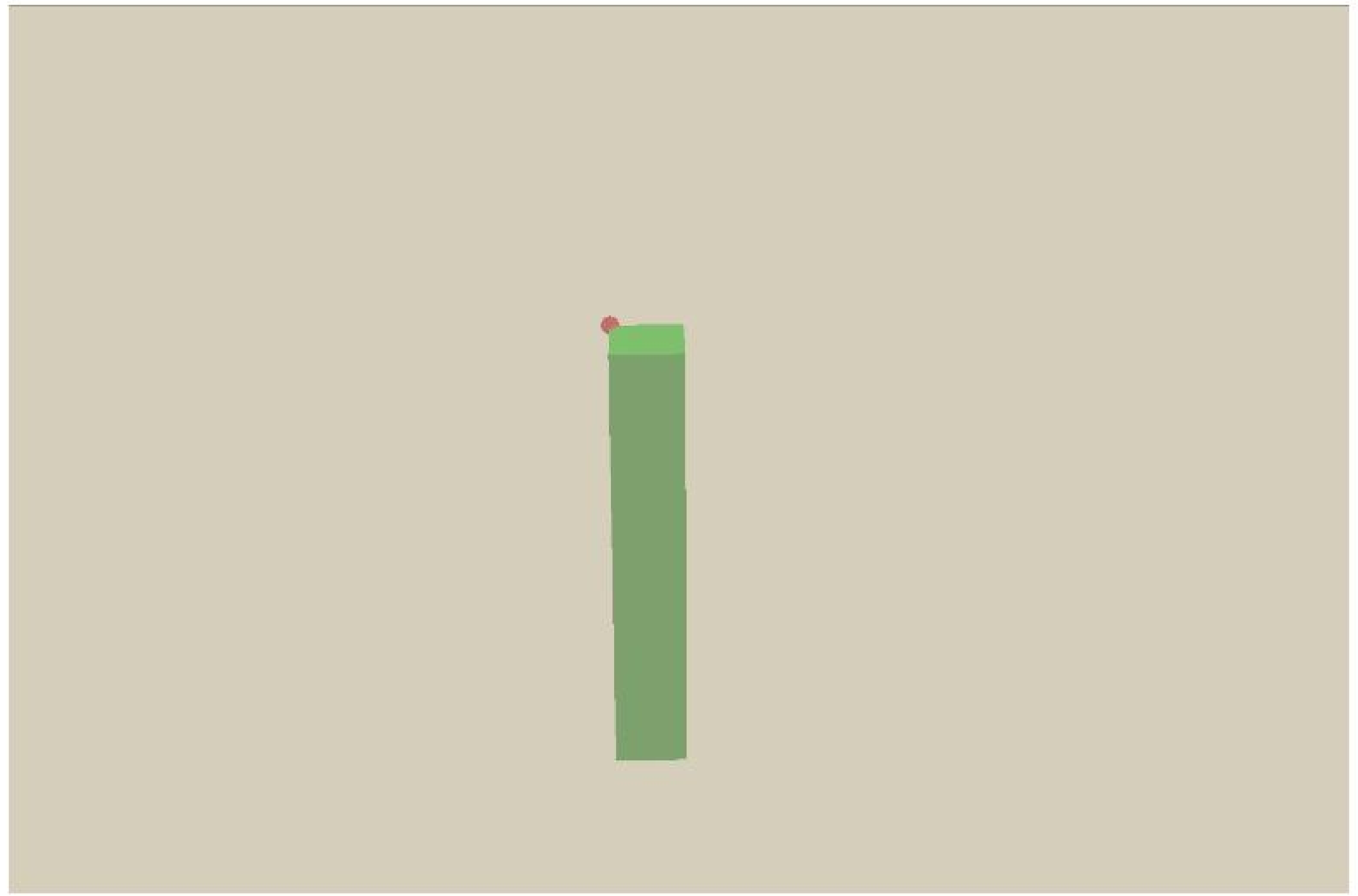
$$\boldsymbol{\tau}^b = \frac{\mathrm{d}\boldsymbol{L}^b}{\mathrm{d}t} = \frac{\mathrm{d}(\boldsymbol{I}^b\boldsymbol{\omega}^b)}{\mathrm{d}t} = \frac{\mathrm{d}\boldsymbol{I}^b}{\mathrm{d}t}\boldsymbol{\omega}^b + \boldsymbol{I}^b\frac{\mathrm{d}\boldsymbol{\omega}^b}{\mathrm{d}t} = \boldsymbol{\omega}^b \times \boldsymbol{I}^b\boldsymbol{\omega}^b + \boldsymbol{I}^b\dot{\boldsymbol{\omega}}^b$$

- We used $\dot{\boldsymbol{I}}^b = [\boldsymbol{\omega}]\boldsymbol{I}^b$ without proof
- An observation (example): Even if there is no torque input, if the object has a non-zero angular velocity $\boldsymbol{\omega}^b$, then it may still have an angular acceleration $\dot{\boldsymbol{\omega}}^b$
  - When $\boldsymbol{\omega}^b \nparallel \boldsymbol{I}^b\boldsymbol{\omega}^b$, i.e., $\boldsymbol{\omega}^b$ is not along the eigenvector of $\boldsymbol{I}^b$, $\boldsymbol{\omega}^b$ will **NOT** keep unchanged
  - $\boldsymbol{\omega}^b$ will not converge ($\dot{\boldsymbol{\omega}}^b$ will never be zero). Its trajectory will form a periodic curve
  - $\boldsymbol{L}^b = \boldsymbol{I}^b\boldsymbol{\omega}^b$ is conserved

Euler Equation

$$\boldsymbol{\tau}^b = \boldsymbol{\omega}^b \times \boldsymbol{I}^b \boldsymbol{\omega}^b + \boldsymbol{I}^b \dot{\boldsymbol{\omega}}^b \qquad \text{(angular motion)}$$





A numerical experiment in SAPIEN for $\boldsymbol{\tau}^b = 0$

(this is illustrative and there are numerical errors)