L6: Dynamics (I)

Hao Su

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Kinematics v.s. Dynamics

- *Kinematics* describes the motion of objects. We have been talking about rigid transformation and derivatives w.r.t. time.
- *Dynamics* describes the cause of motion. We will talk about mass, energy, momentum, and force.
- The basic law of dynamics, Newton's Law, describes the motion of a point mass:

$$\boldsymbol{f} = m\boldsymbol{a}$$

- But there are caveats that you may not be aware of.

Kinematics v.s. Dynamics

- We start from point mass dynamics and will move on to rigid body dynamics.
- We will provide certain proofs but not all (many are very tricky and lengthy).

A Tale of Three Frames

Concepts

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- **Observer's Frame**:
  - When we record any motion, we choose the observer's frame $\mathcal{F}_o$, so that every point would have a coordinate and every vector will have a direction and length.
  - For our symbols, this is on the superscript.
  - If the frame is moving (e.g., taken to be the body frame), when recording motions, we first *clone a version* of this frame and *keep it static* for recording.

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- **Reference Frame**:
  - When recording the movement of objects, we introduce a reference frame so that the notion of movement is *relative to* this frame.

Some Notes on Reference Frame

- Reference Frame:
  - When recording the movement of objects, we introduce a reference frame so that the notion of movement is *relative to* this frame.
- We have not discussed this frame much in developing robot **kinematics theories**.

In dynamics, the choice of reference frame is not arbitrary!

Recording a *Relative* Velocity

- We introduce $s(t)$ to denote a reference frame which may be moving.
- Then we denote the relative velocity as below:
- Relative velocity for a point mass
    - $\boldsymbol{v}^o_{s(t)\to b(t)} = \boldsymbol{v}^o_{o\to b(t)} - \boldsymbol{v}^o_{o\to s(t)}$
- Relative velocity for rigid body
    - $\boldsymbol{\xi}^o_{s(t)\to b(t)} = \boldsymbol{\xi}^o_{b(t)} - \boldsymbol{\xi}^o_{s(t)}$
- Consistency

$$\boldsymbol{v}^o_{s(t)\to b(t)} = \boldsymbol{\xi}^o_{s(t)\to b(t)} \boldsymbol{p}^o$$

where p^o is a point observed in $\mathcal{F}_o$

Inertia Frame

- Inertia frame refers to the choice of the *reference frame*.
- Only in an inertia frame can Newton's law be written as $\boldsymbol{f} = m\boldsymbol{a}$.
- Definition of Inertia frame:
  - Where the law of inertia (Newton's First Law) is satisfied.
  - Any free motion has a constant magnitude and direction.
- A clear notion of Newton's Second Law:

$$\boldsymbol{f}^o = m\boldsymbol{a}^o_{s(t)\to b(t)}$$

where $s(t)$ is an inertia frame (o is static).

Fictitious Force

- What if the reference frame is not an inertia frame?
- Assume we have two moving frames, $\mathcal{F}_{s(t)}$ and $\mathcal{F}_{b(t)}$
  - e.g., the earth and an object sitting on the earth
- We are interested in how the force $\boldsymbol{f}^o$ affects the relative acceleration $\boldsymbol{a}^o_{s(t)\to b(t)}$

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- Some intuition that $\boldsymbol{f}^o \neq m\boldsymbol{a}^o_{s(t)\to b(t)}$
  - Since $\mathcal{F}_{s(t)}$ is moving with an angular velocity, any object $b(t)$ moving along with it must also have an acceleration to gain the same angular velocity.
  - Computation shows that some additional force will be consumed to maintain the relative velocity of $b(t)$ against $s(t)$.

Fictitious Force

- Computing $\boldsymbol{f}^o = \mathrm{d}(m\boldsymbol{v}^o_{s'\to b(t)})/\mathrm{d}t$ (note: s' is chosen to be an inertia frame), and we have

$$\boldsymbol{f}^o - m\frac{\mathrm{d}\boldsymbol{\omega}^o}{\mathrm{d}t} \times \boldsymbol{r}^o - 2m\boldsymbol{\omega}^o \times \boldsymbol{v}^o - m\boldsymbol{\omega}^o \times (\boldsymbol{\omega}^o \times \boldsymbol{r}^o) = m\boldsymbol{a}^o$$

  where
  - $\boldsymbol{f}^o$: the physical forces acting on the object
  - $\boldsymbol{\omega}^o := \boldsymbol{\omega}^s_{s'\to s(t)}$
  - $\boldsymbol{v}^o := \boldsymbol{v}^o_{s(t)\to b(t)}$
  - $\boldsymbol{r}^o := \boldsymbol{r}^o_{s(t)\to b(t)}$
  - $\boldsymbol{a}^o := \boldsymbol{a}^o_{s(t)\to b(t)}$

Fictitious Force

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- Euler force: $-m\frac{\mathrm{d}\boldsymbol{\omega}^o}{\mathrm{d}t} \times \boldsymbol{r}^o$
- Centrifugal force: $-m\boldsymbol{\omega}^o \times (\boldsymbol{\omega}^o \times \boldsymbol{r}^o)$
- Coriolis force: $-2m\boldsymbol{\omega}^o \times \boldsymbol{v}^o$

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$$f = ma$$

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