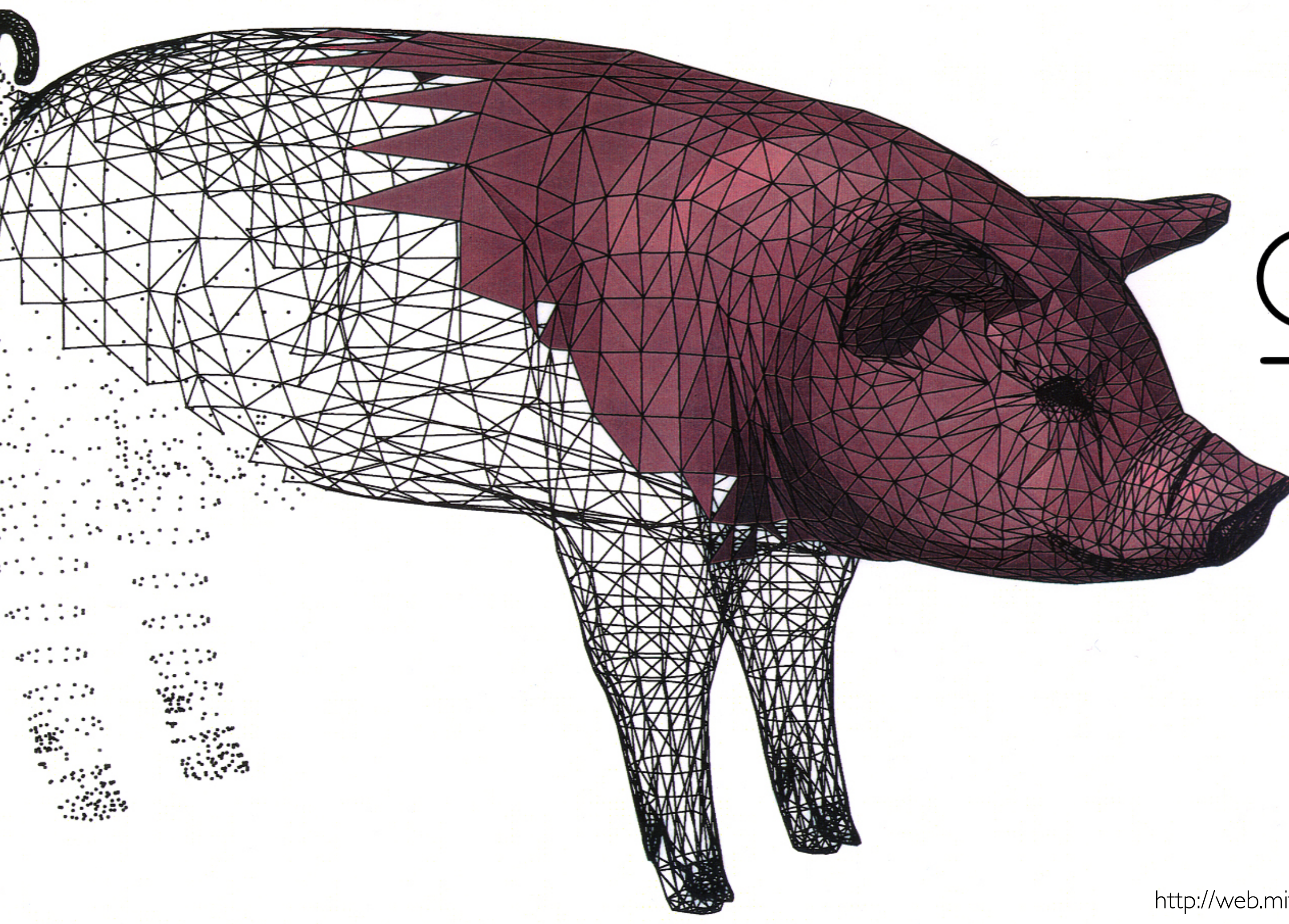
Office Hour

- Check Piazza

L2: Surfaces

Hao Su

Our Focus Today: Surface

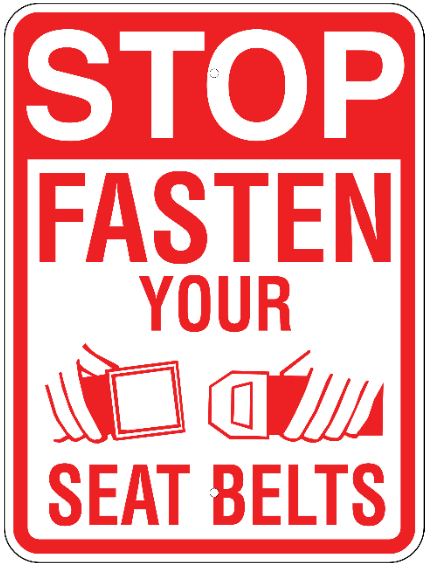


$\subseteq \mathbb{R}^3$

http://web.mit.edu/manoli/crust/www/slides/piggy.jpg

Agenda

- Parameterized Surface
- Manifold
- Differential Map
- Curvature
- Principal Curvature



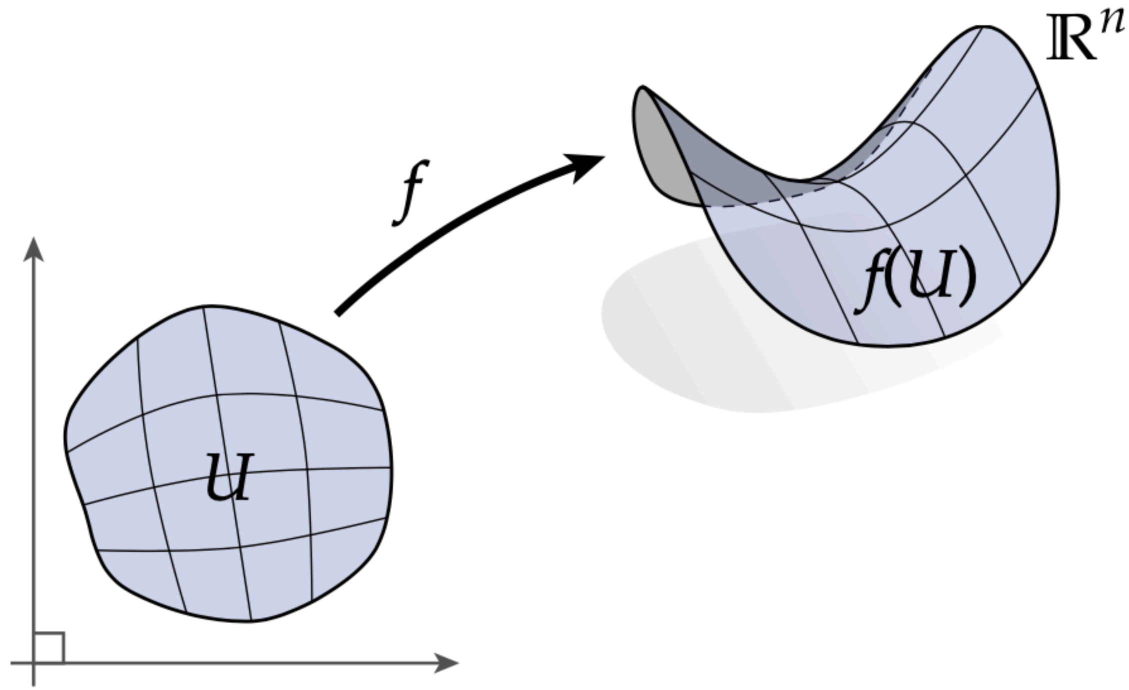
Lots of (sloppy) math!

Parameterized Surface

Parametrized Surface

A **parameterized surface** is a map from a two-dimensional region $U \subset \mathbb{R}^2$ into $\mathbb{R}^n$

$$f : U \to \mathbb{R}^n$$



The set of points $f(U)$ is called the **image** of the parameterization.

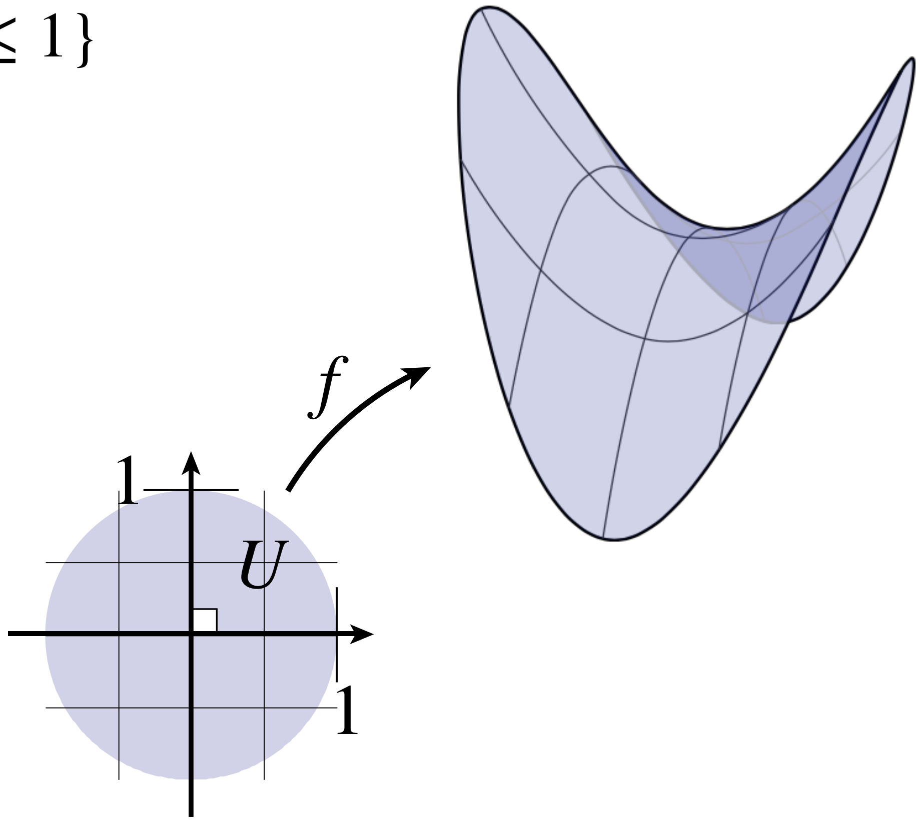
Image from Wikipedia

Example

- Example: We can express a *saddle as a parameterized* surface*:*

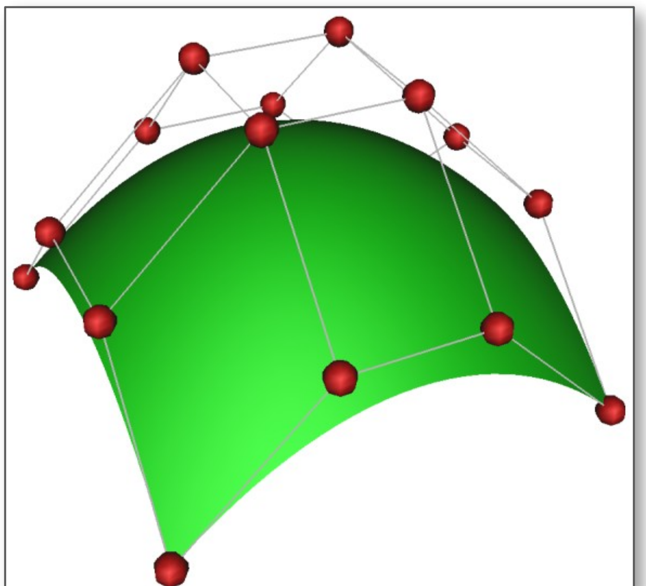
$$U := \{(u, v) \in \mathbb{R}^2 : u^2 + v^2 \leq 1\}$$

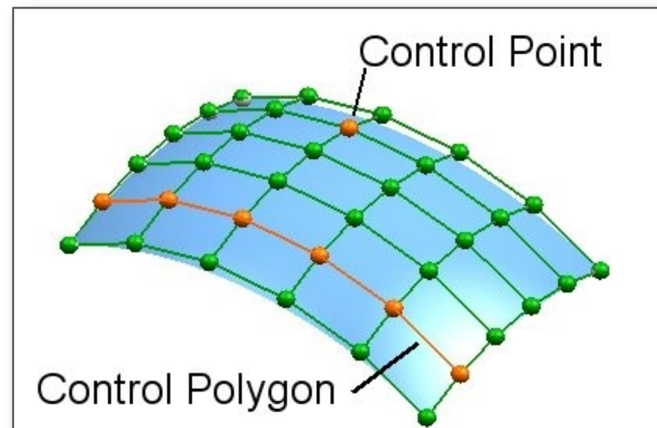
$$f(u, v) = [u, v, u^2 - v^2]^T$$



Application: Bezier Surface, Spline Surface

- Smoothly "interpolate" between *a set of* points P_i

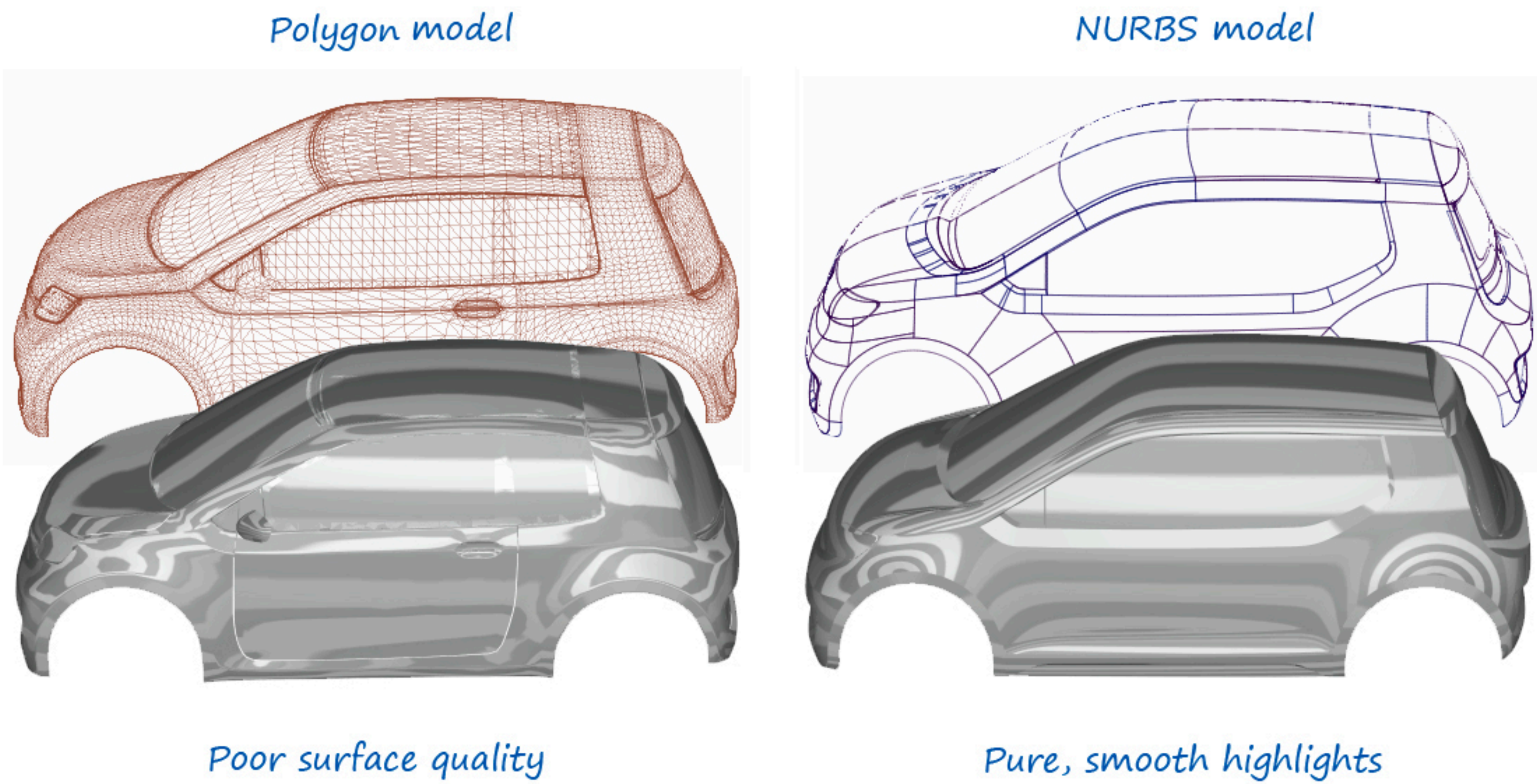




$$s(u,v) = \sum_{i=0}^{m} \sum_{j=0}^{n} \mathbf{p}_{i,j} B_i^m(u) B_j^n(v)$$

Application: Bezier Surface, Spline Surface

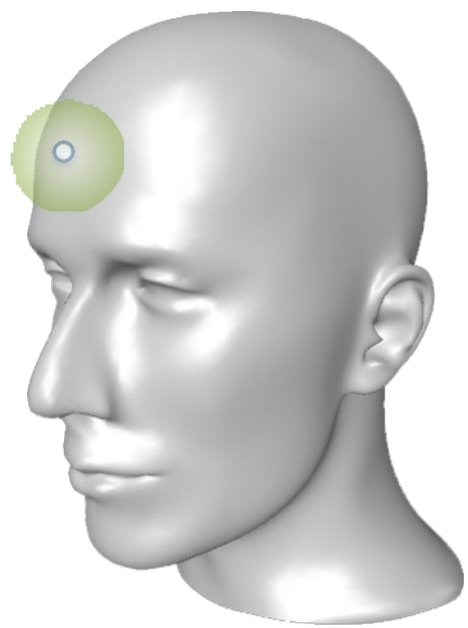
Widely used in design industry (e.g., car modeling)



(Differentiable) Manifold

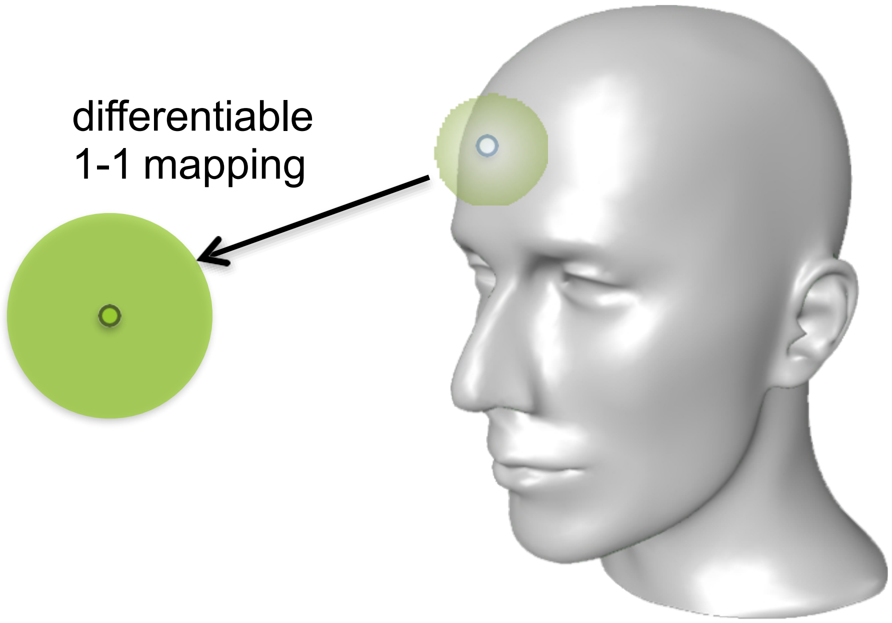
Smoothness as a Local Property

- Things that can be discovered by local observation: point + neighborhood



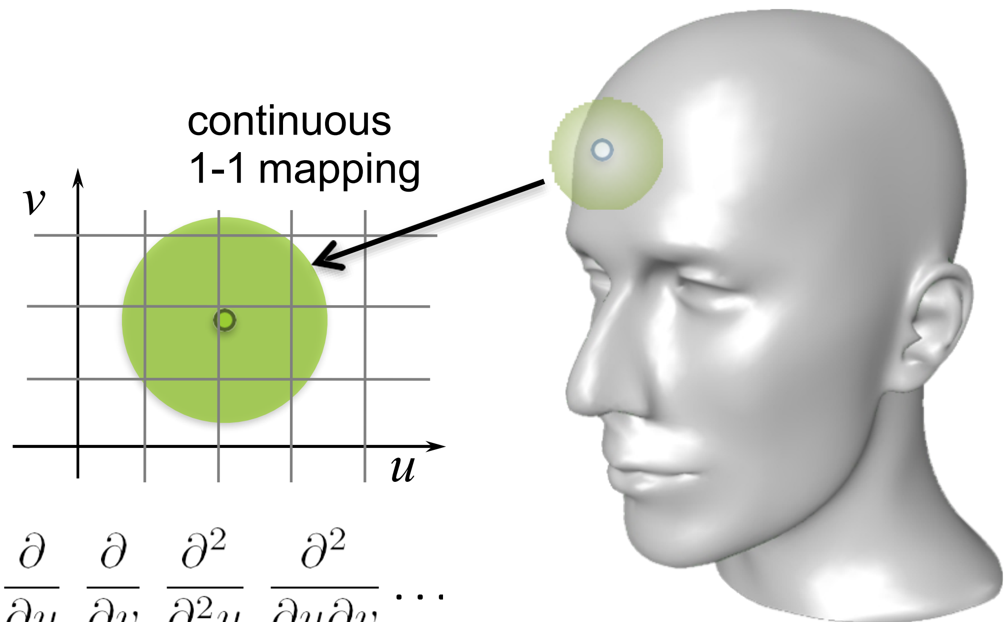
Local Smoothness

- Things that can be discovered by local observation: point + neighborhood



Local to Global

- Things that can be discovered by local observation: point + neighborhood

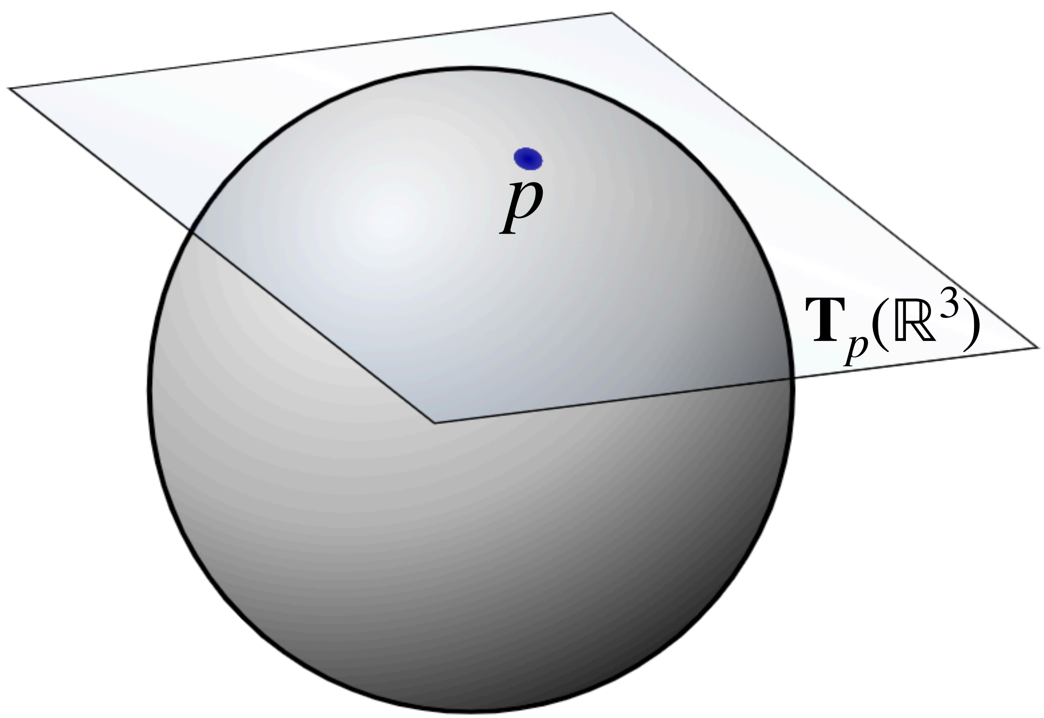


$$\frac{\partial}{\partial u} \quad \frac{\partial}{\partial v} \quad \frac{\partial^2}{\partial^2 u} \quad \frac{\partial^2}{\partial u \partial v} \ \ldots$$

Tangents, normals, curvatures, curve angles, distances

Tangent Plane

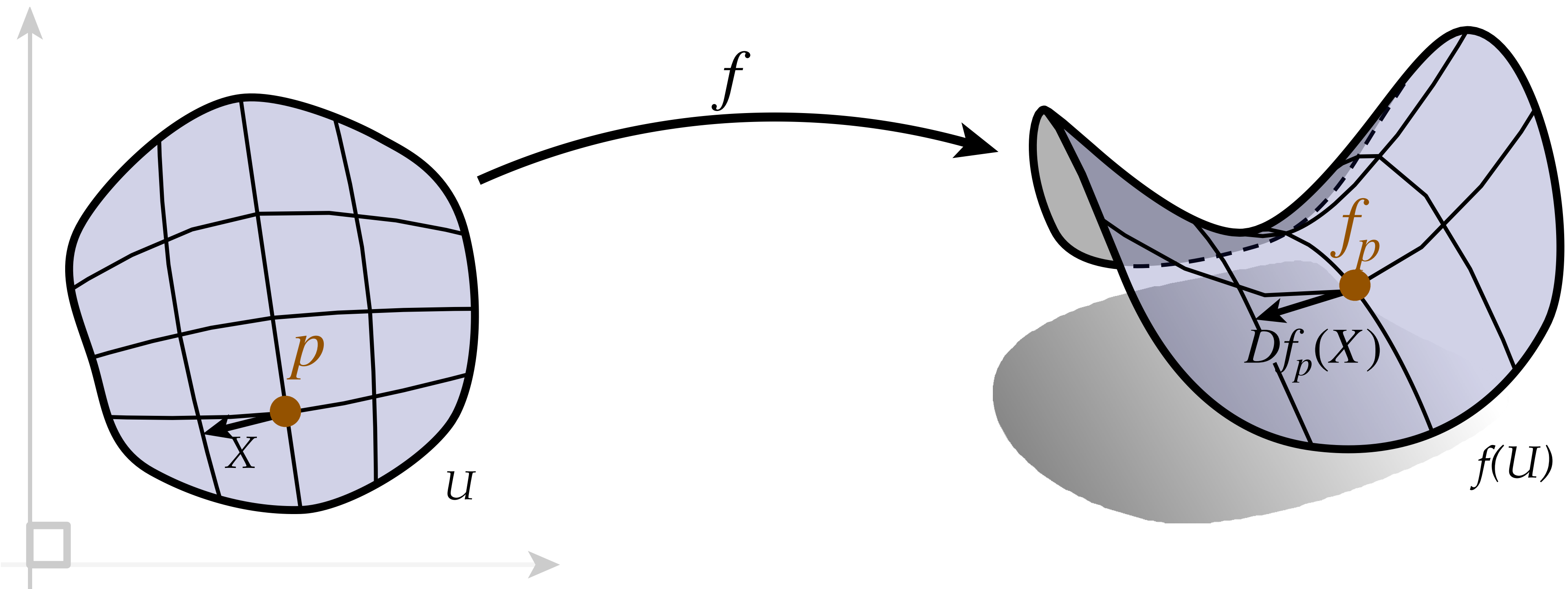
- One can attach to every point p a tangent plane $\mathbf{T}_p$
- Intuitively, it contains the possible directions in which one can tangentially pass through p.



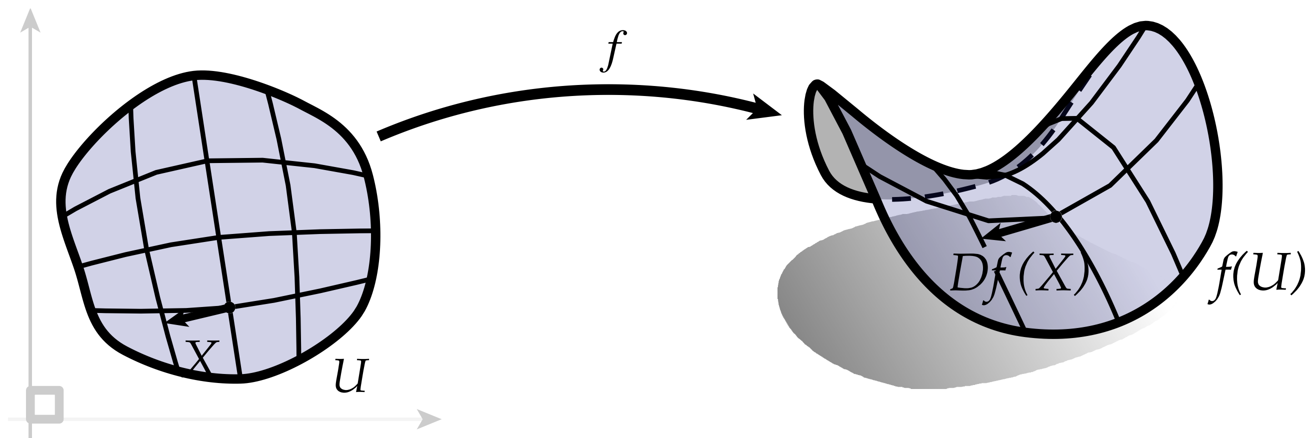
Differential Map

Differential of a Surface

- Relate the movement of point in the domain and on the surface



Differential of a Surface

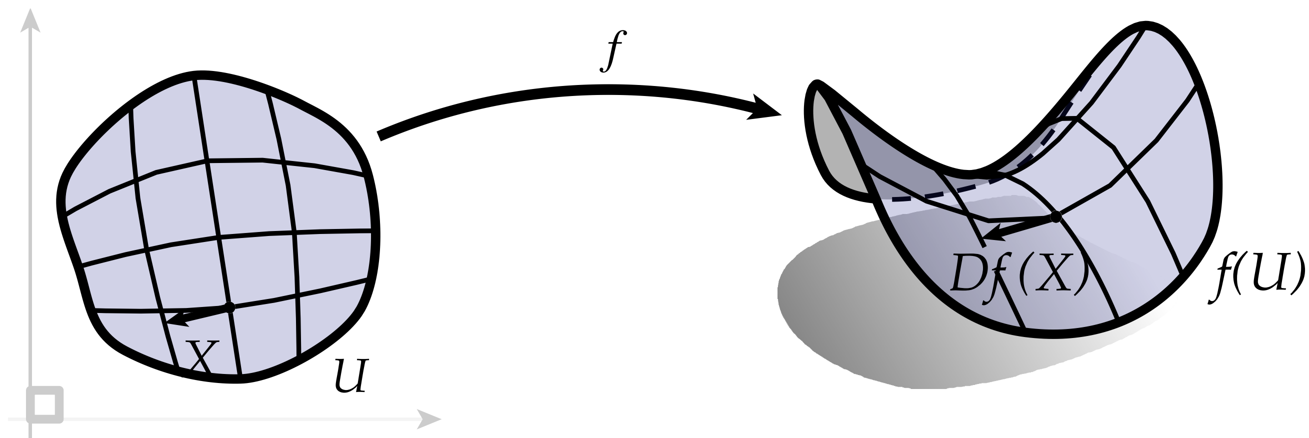


Total differential: $df = \frac{\partial f}{\partial u} du + \frac{\partial f}{\partial v} dv \Longrightarrow \Delta f \approx \frac{\partial f}{\partial u} \Delta u + \frac{\partial f}{\partial v} \Delta v$

If point $p \in \mathbb{R}^2$ moves along vector $X = [u, v]^T$ by ϵ, the movement of f_p is:

$$\Delta f_p \approx \frac{\partial f}{\partial u}(\epsilon u) + \frac{\partial f}{\partial v}(\epsilon v) = \epsilon \left[\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v} \right] \begin{bmatrix} u \\ v \end{bmatrix}$$

Differential of a Surface



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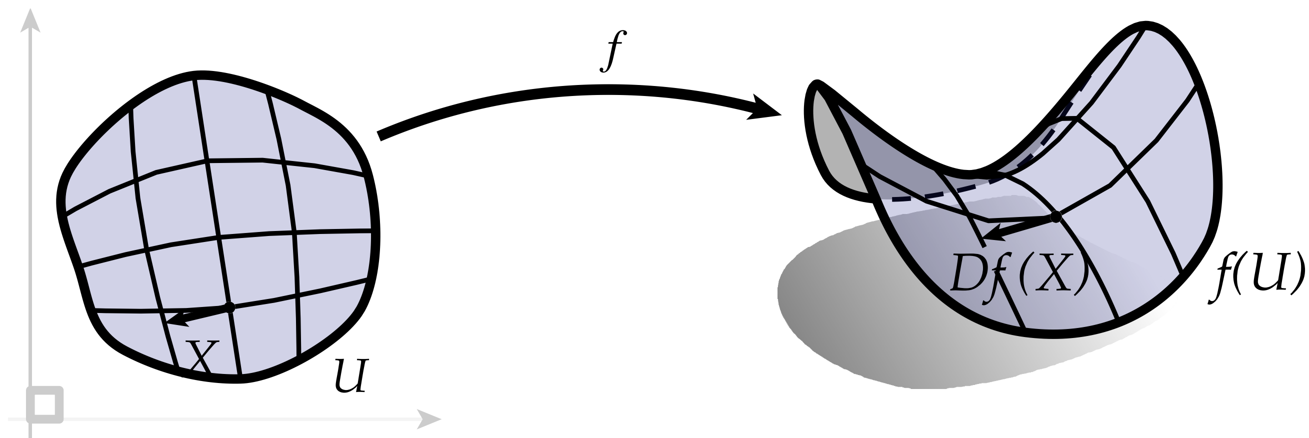
$$Df_p := \left[\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v}\right] \in \mathbb{R}^{3\times 2}$$

Differential of a Surface



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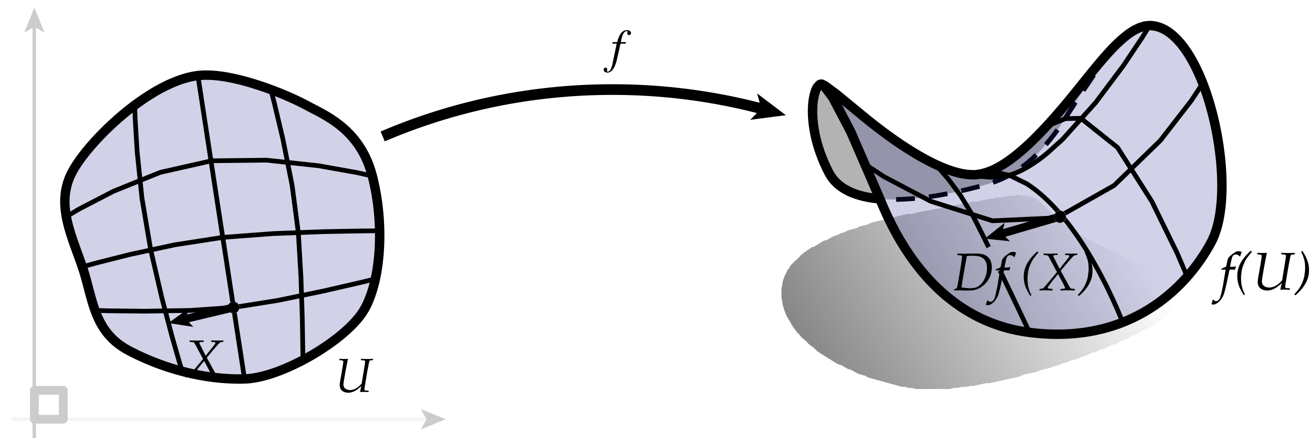
$Df_p := \left[\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v} \right] \in \mathbb{R}^{3\times 2}$

Df_p: differential (Jacobian)
a linear map.

Differential of a Surface



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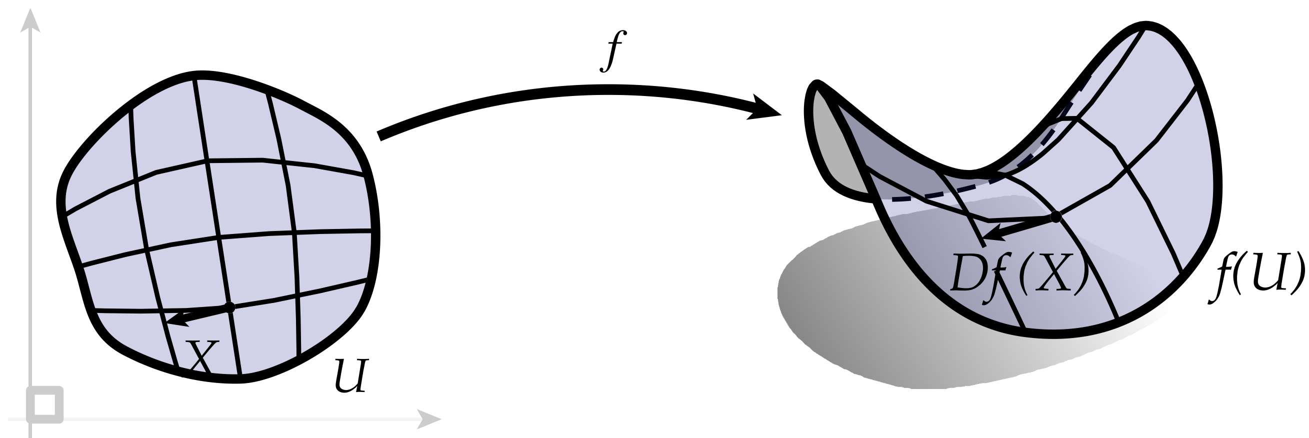
$$Df_p := \left[\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v}\right] \in \mathbb{R}^{3 \times 2}$$

velocity in 2D domain

Differential of a Surface



Total differential: $df = \frac{\partial f}{\partial u} du + \frac{\partial f}{\partial v} dv$

If point $p \in \mathbb{R}^2$ moves with velocity $X = [u, v]^T$ by ϵ, the movement of f_p is:

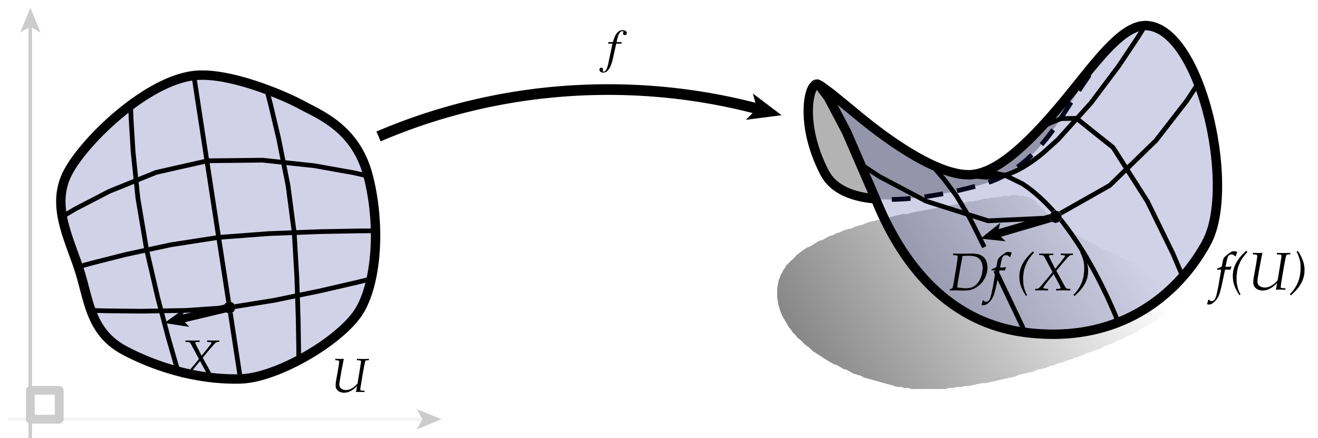
velocity in 3D space

$$\Delta f_p \approx \frac{\partial f}{\partial u}(\epsilon u) + \frac{\partial f}{\partial v}(\epsilon v) = \epsilon \left[\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v} \right] \begin{bmatrix} u \\ v \end{bmatrix} = \epsilon [Df_p] X$$

$$Df_p := \left[\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v} \right] \in \mathbb{R}^{3 \times 2}$$

velocity in 2D domain

Differential of a Surface



Intuitively, the *differential* of a parameterized surface tells us how tangent vectors on the domain get mapped to tangent vectors in space:

$$\Delta f_p \approx \frac{\partial f}{\partial u}(\epsilon u) + \frac{\partial f}{\partial v}(\epsilon v) = \epsilon \left[\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v} \right] \begin{bmatrix} u \\ v \end{bmatrix} = \epsilon [Df_p] X$$

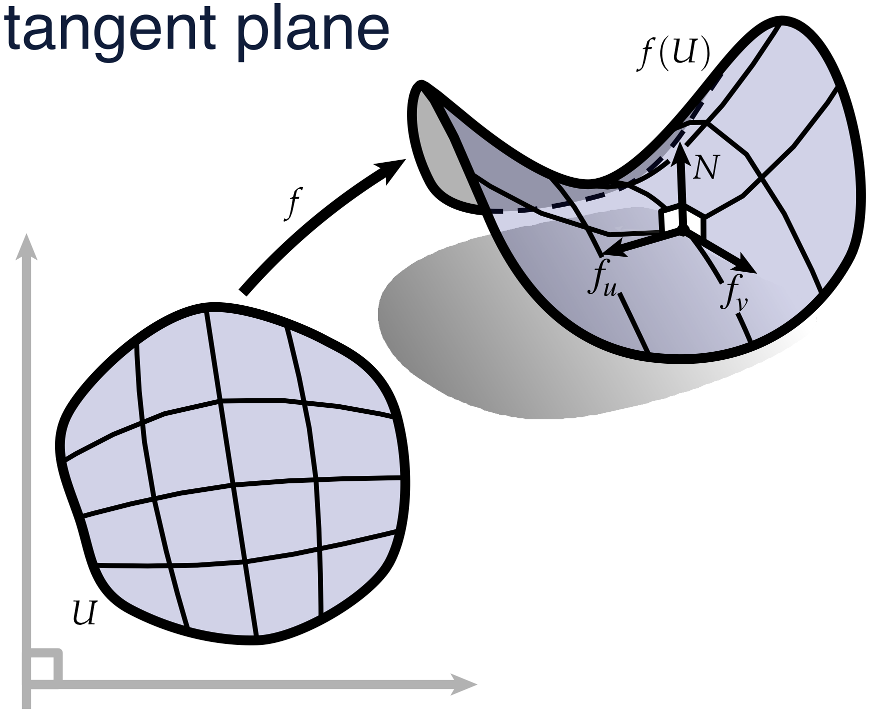
Tangent Plane

$$\Delta f_p \approx \frac{\partial f}{\partial u}(\epsilon u) + \frac{\partial f}{\partial v}(\epsilon v) = \epsilon \left[\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v}\right] \begin{bmatrix} u \\ v \end{bmatrix}$$

$\left[\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v}\right] \begin{bmatrix} u \\ v \end{bmatrix}$ is a vector in 3D tangent plane

Tangent plane at point $f(u, v)$ is spanned by

$$f_u = \frac{\partial f}{\partial u}, \; f_v = \frac{\partial f}{\partial v}$$

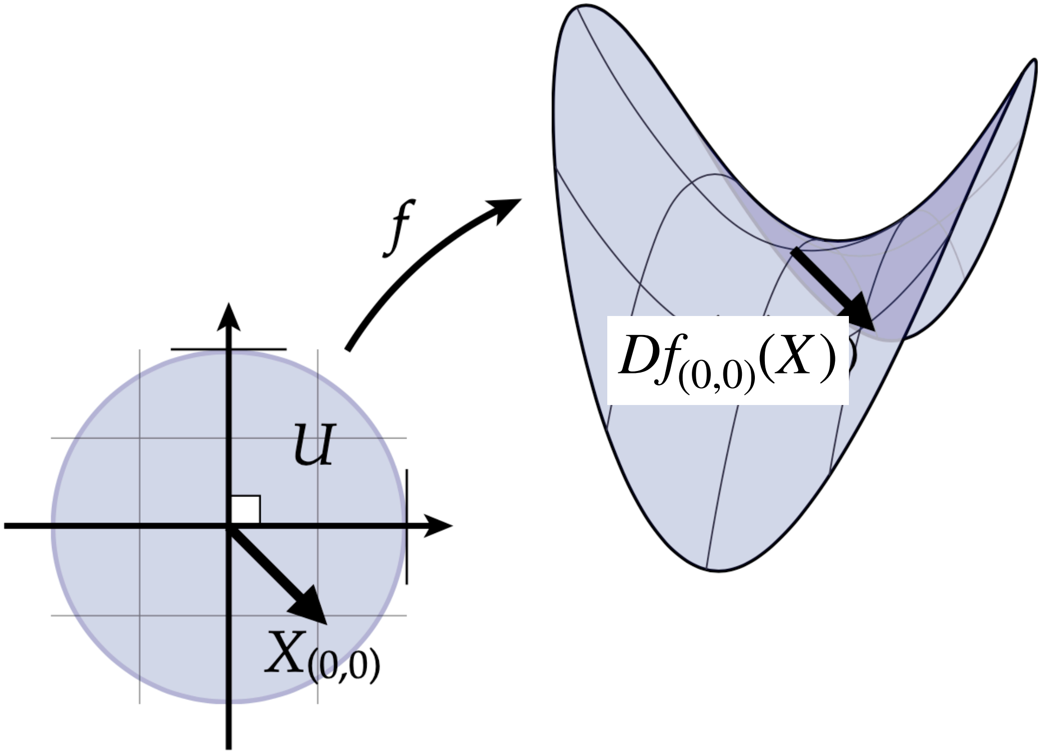


These vectors don't have to be orthogonal

An Example

$$f(u,v) = [u, v, u^2 - v^2]^T$$

$$Df_p = \begin{bmatrix} \partial f_1/\partial u & \partial f_1/\partial v \\ \partial f_2/\partial u & \partial f_2/\partial v \\ \partial f_3/\partial u & \partial f_3/\partial v \end{bmatrix} =$$



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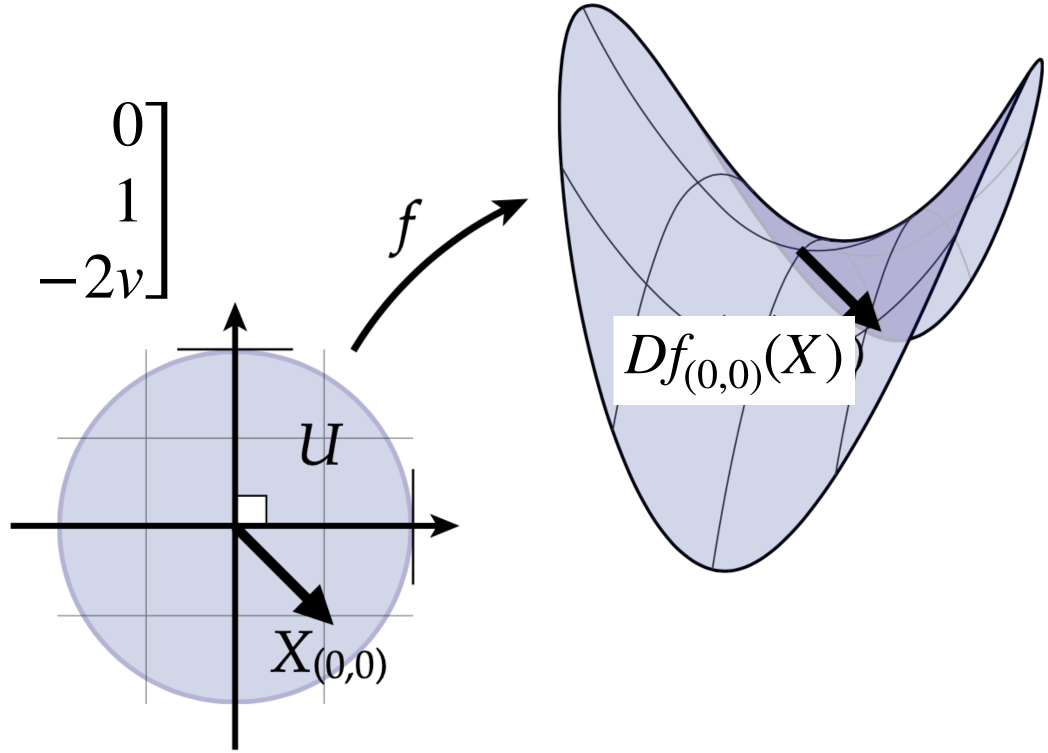

An Example

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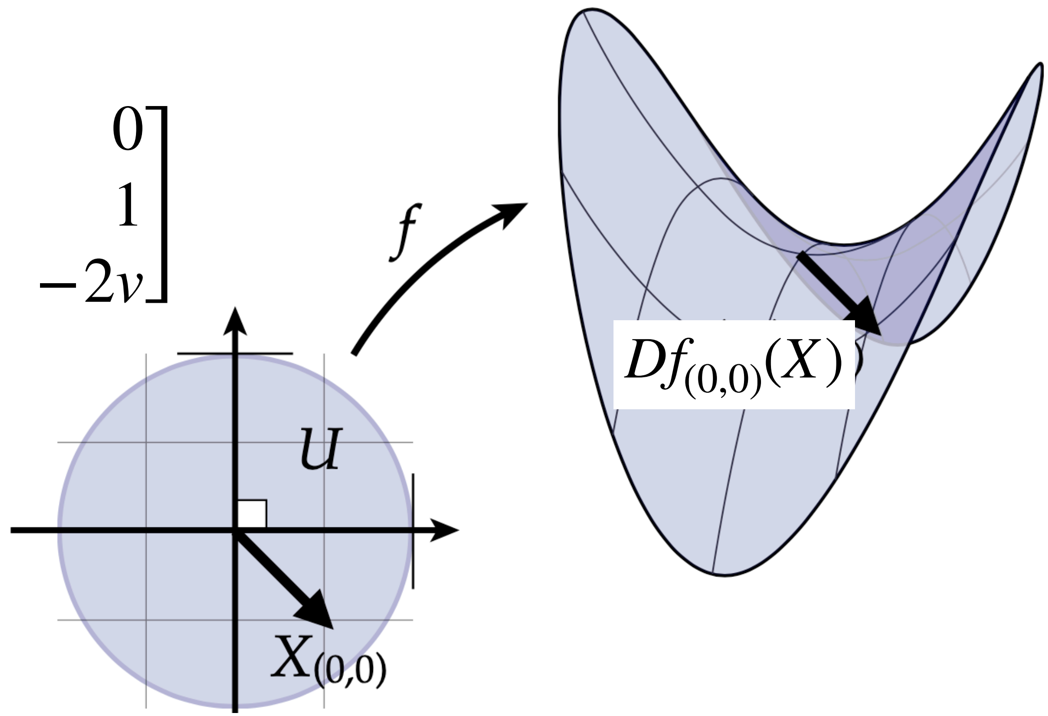
$$X := \frac{3}{4}[1, -1]^T$$

$$Df(X) =$$



An Example

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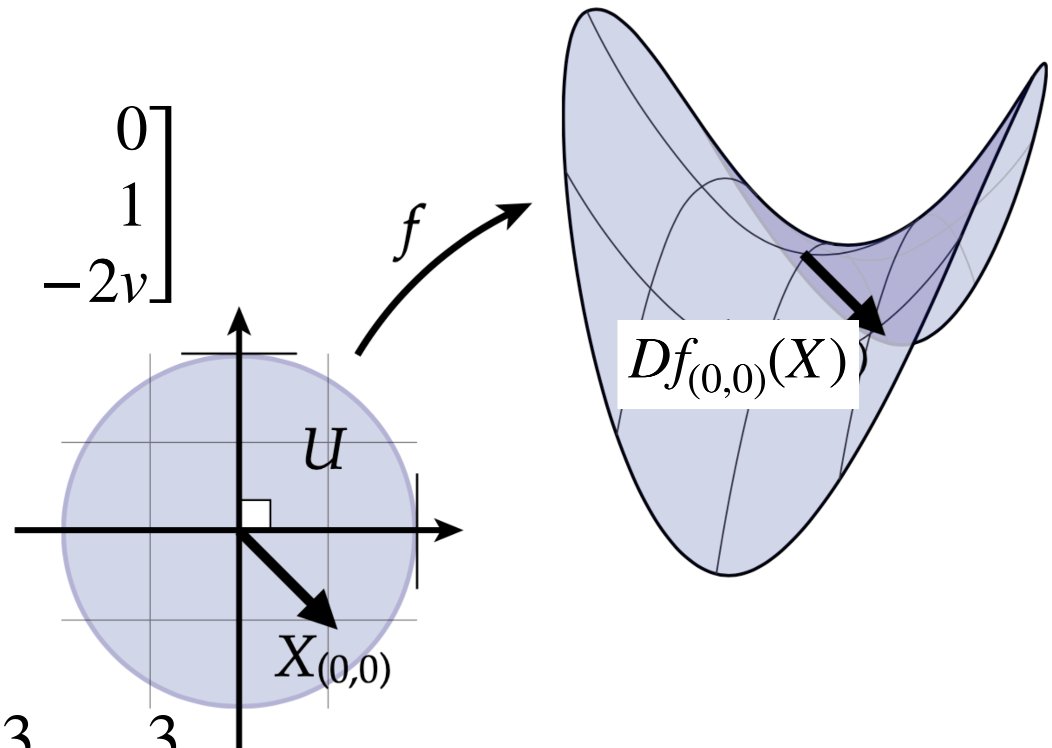
$$Df_p = \begin{bmatrix} \partial f_1/\partial u & \partial f_1/\partial v \\ \partial f_2/\partial u & \partial f_2/\partial v \\ \partial f_3/\partial u & \partial f_3/\partial v \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2u & -2v \end{bmatrix}$$

$$X := \frac{3}{4}[1, -1]^T$$

$$Df(X) = \frac{3}{4}[1, -1, 2(u+v)]^T$$



e.g., at $u = v = 0 : Df(X) = [\frac{3}{4}, -\frac{3}{4}, 0]^T$

An Example

$$f(u,v) = [u, v, u^2 - v^2]^T$$

$$Df_p = \begin{bmatrix} \partial f_1/\partial u & \partial f_1/\partial v \\ \partial f_2/\partial u & \partial f_2/\partial v \\ \partial f_3/\partial u & \partial f_3/\partial v \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2u & -2v \end{bmatrix}$$

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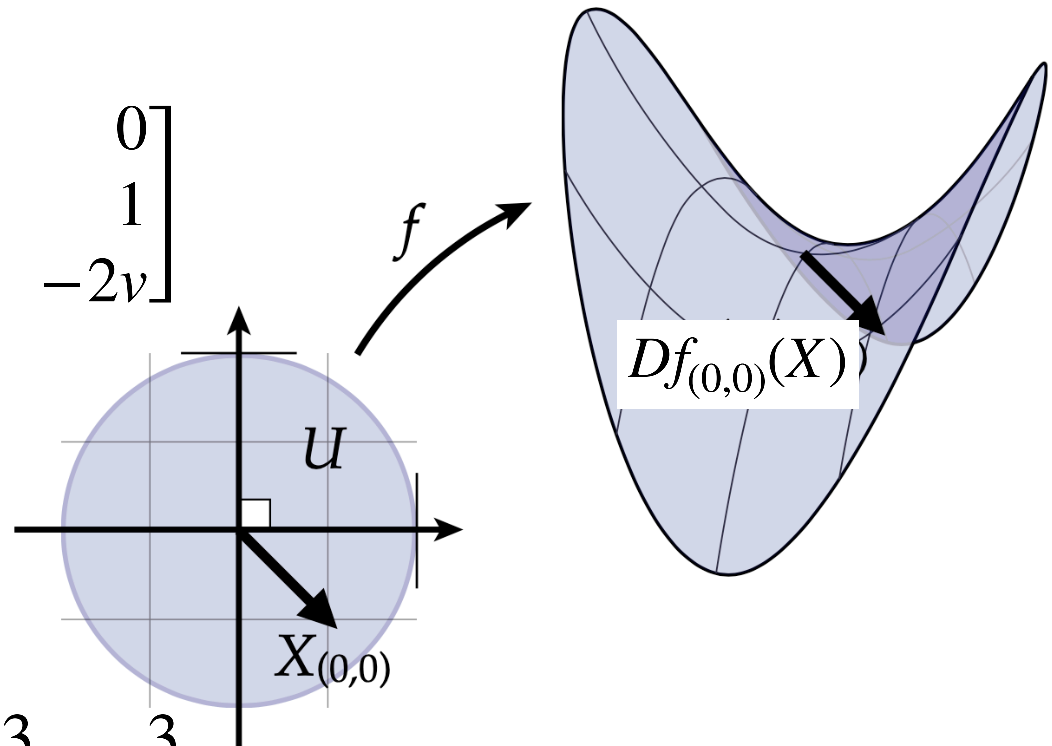

e.g., at $u = v = 0$: $Df(X) = [\frac{3}{4}, -\frac{3}{4}, 0]^T$

at $u = v = 1$, tangent space is spanned by

An Example

$$f(u,v) = [u, v, u^2 - v^2]^T$$

$$Df_p = \begin{bmatrix} \partial f_1/\partial u & \partial f_1/\partial v \\ \partial f_2/\partial u & \partial f_2/\partial v \\ \partial f_3/\partial u & \partial f_3/\partial v \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2u & -2v \end{bmatrix}$$

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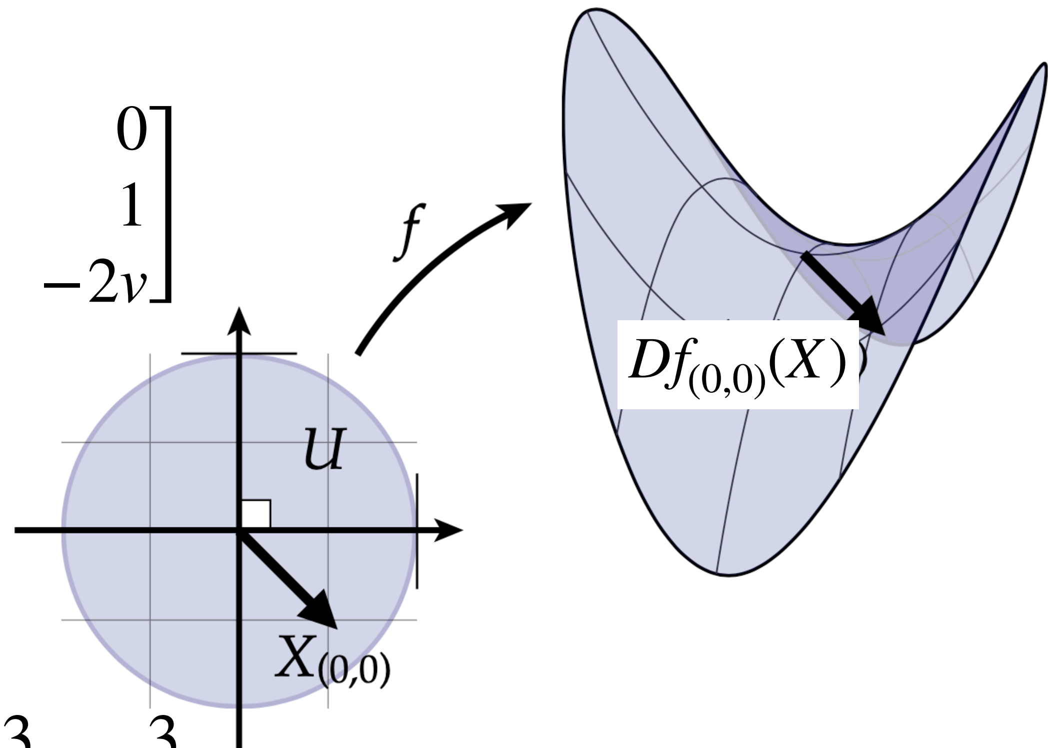


e.g., at $u = v = 0$: $Df(X) = [\frac{3}{4}, -\frac{3}{4}, 0]^T$

at $u = v = 1$, tangent space is spanned by $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$.

Summary of Differential Map

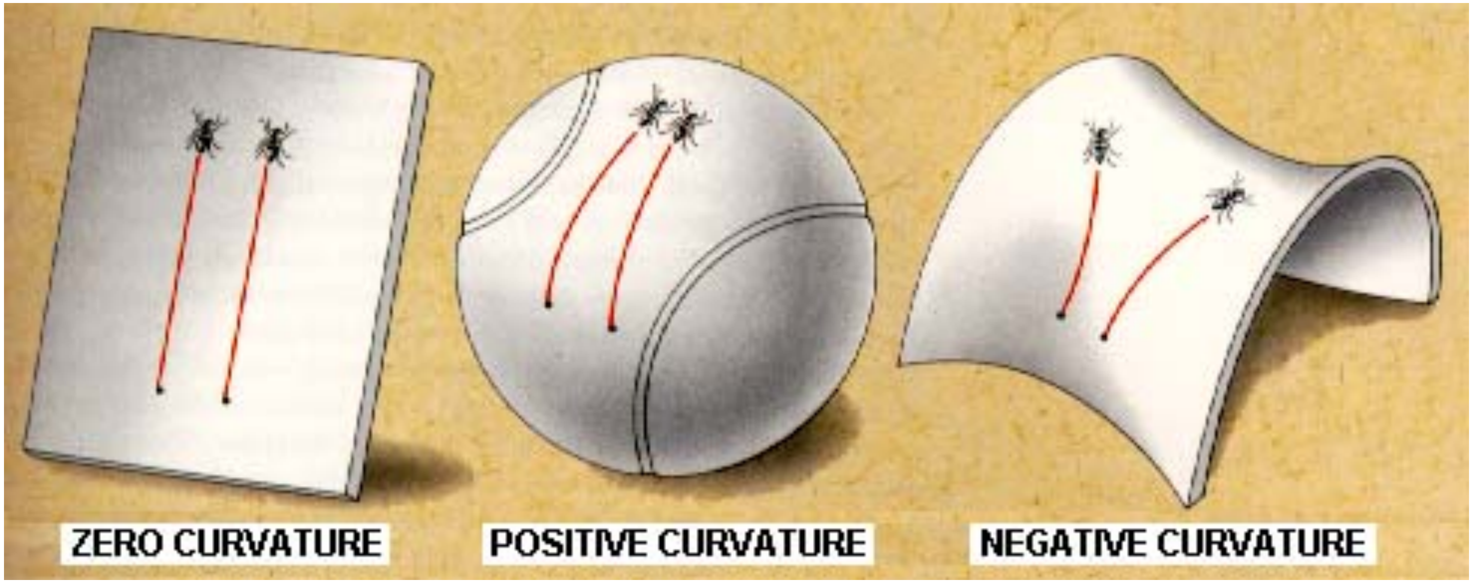
- Tells us the velocity of point in 3D when the parameter changes in 2D
- Maps a vector in the tangent space of the domain to the tangent space of the surface
- Allows us to construct the bases of tangent plane
- Is a linear map

$$Df_p : \mathbf{T}_p(\mathbb{R}^2) \rightarrow \mathbf{T}_{f(p)}(\mathbb{R}^3)$$

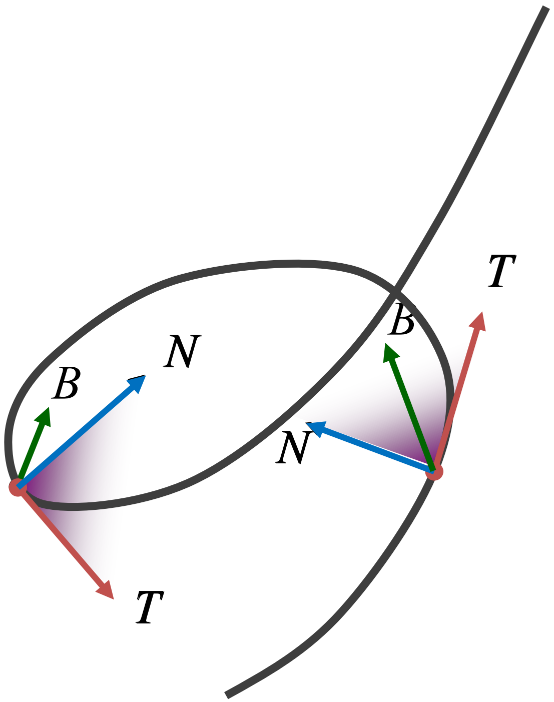
Curvature

Goal

Quantify how a surface **bends**.



Recall: **Curvature of Curves**



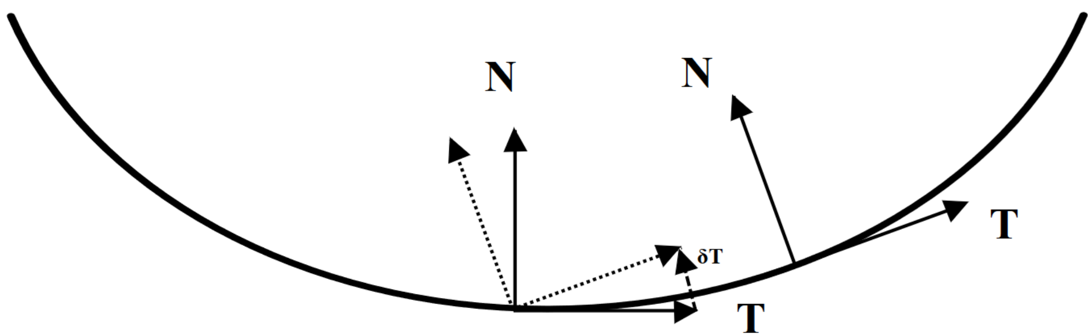
Theorem:

Curvature and torsion determine geometry of a curve up to rigid motion.

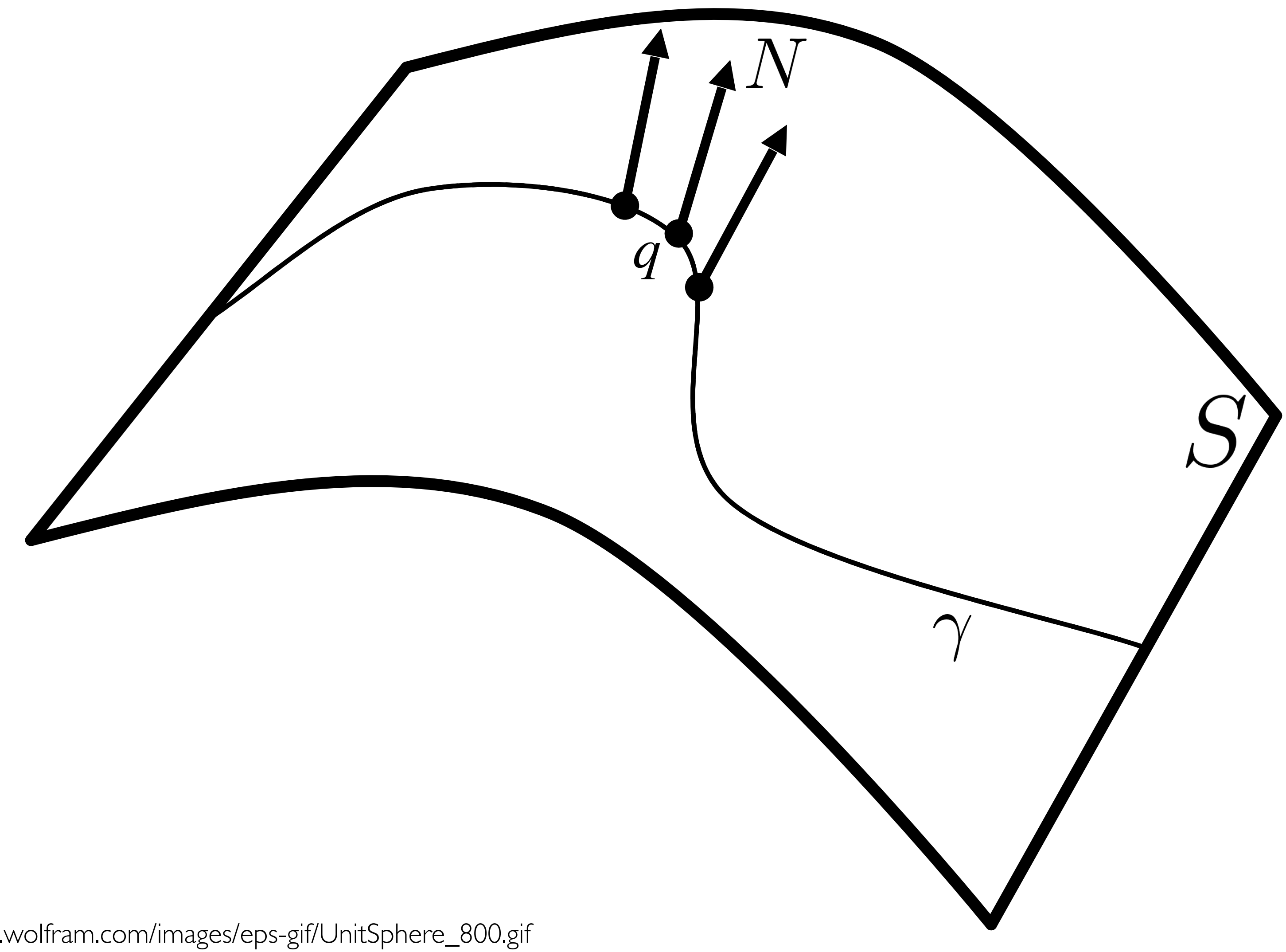
Q:

Can curvature/torsion of a curve help us understand surfaces?

Curves: Change of Normal Describes Curve Bending



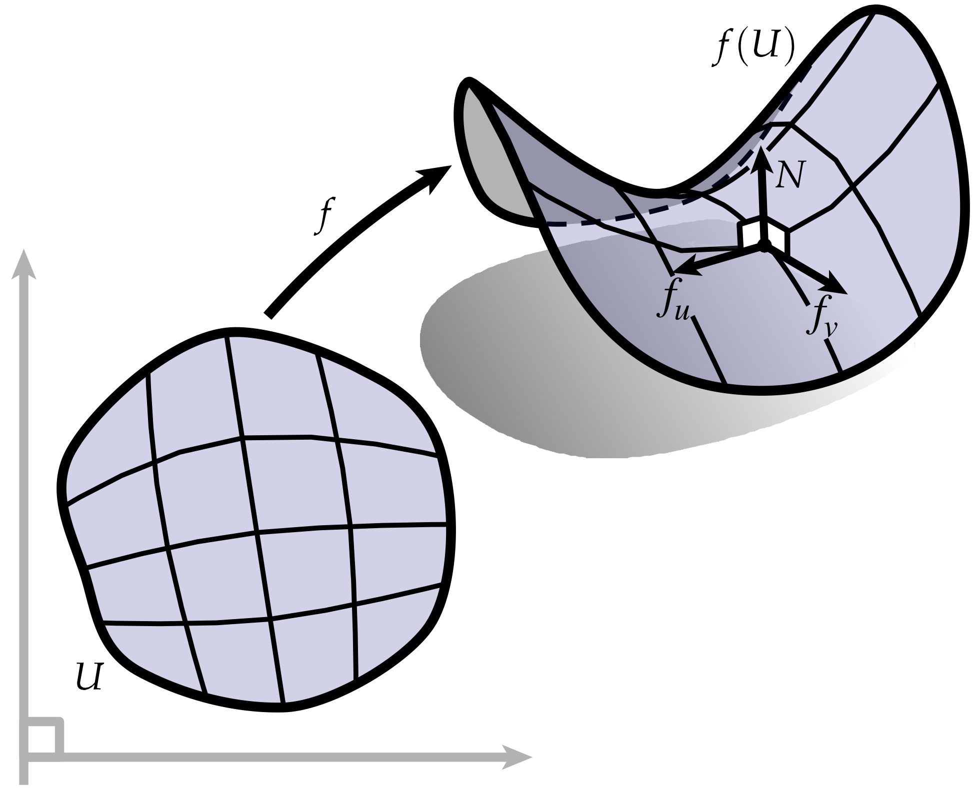
Surfaces: Change of Normal Describes Surface Bending



http://mathworld.wolfram.com/images/eps-gif/UnitSphere_800.gif

Surface Normals

$$f_u := \frac{\partial f}{\partial u},\; f_v := \frac{\partial f}{\partial v}$$



Surface normal:

$$N(u,v) = \frac{f_u \times f_v}{\|f_u \times f_v\|}$$

N also as a function of u, v

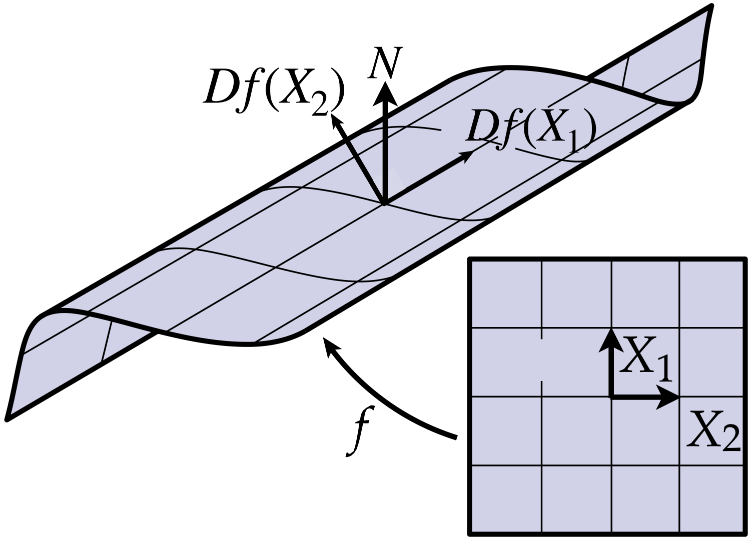
Example

Consider a nonstandard parameterization of the cylinder (sheared along z):

$$f(u, v) := [\cos(u), \sin(u), u + v]^T$$

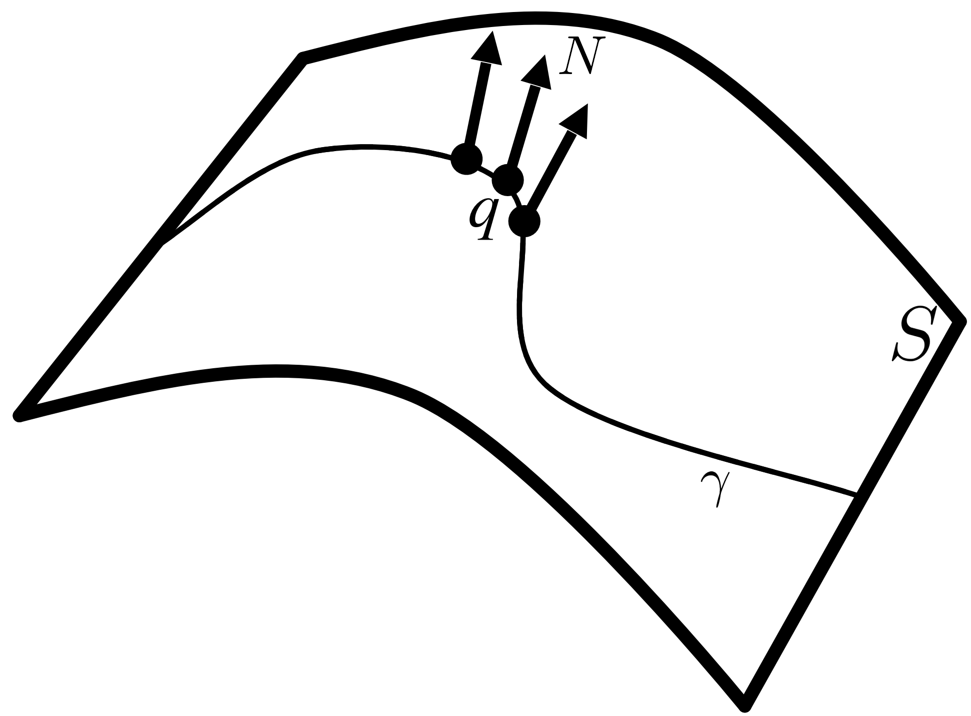
$$Df = \begin{bmatrix} -\sin(u) & 0 \\ \cos(u) & 0 \\ 1 & 1 \end{bmatrix}$$

$$N = \begin{bmatrix} -\sin(u) \\ \cos(u) \\ 1 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \cos(u) \\ \sin(u) \\ 0 \end{bmatrix}$$



Measure the Change of Normal

Assume q moves along a curve γ parameterized by arc-length: $q = \gamma(s)$, and the normal is $N(s)$ with unit norm



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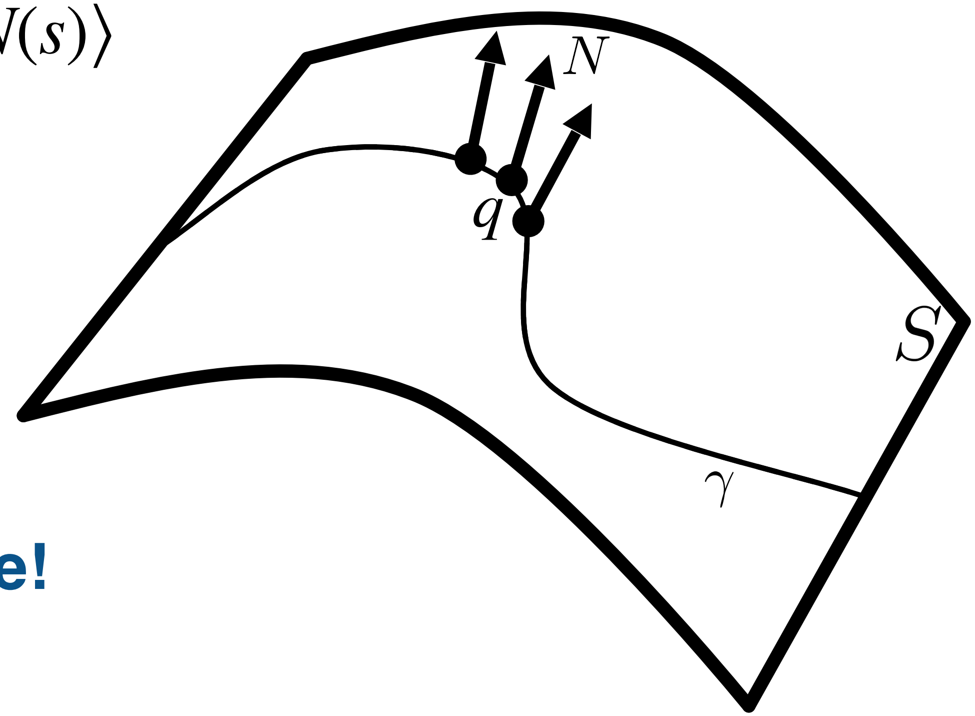
Measure the Change of Normal

Assume q moves along a curve γ parameterized by arc-length: $q = \gamma(s)$, and the normal is $N(s)$ with unit norm

$$0 \equiv \frac{\mathrm{d}}{\mathrm{d}s}\langle N(s), N(s)\rangle = 2\langle \dot{N}(s), N(s)\rangle$$

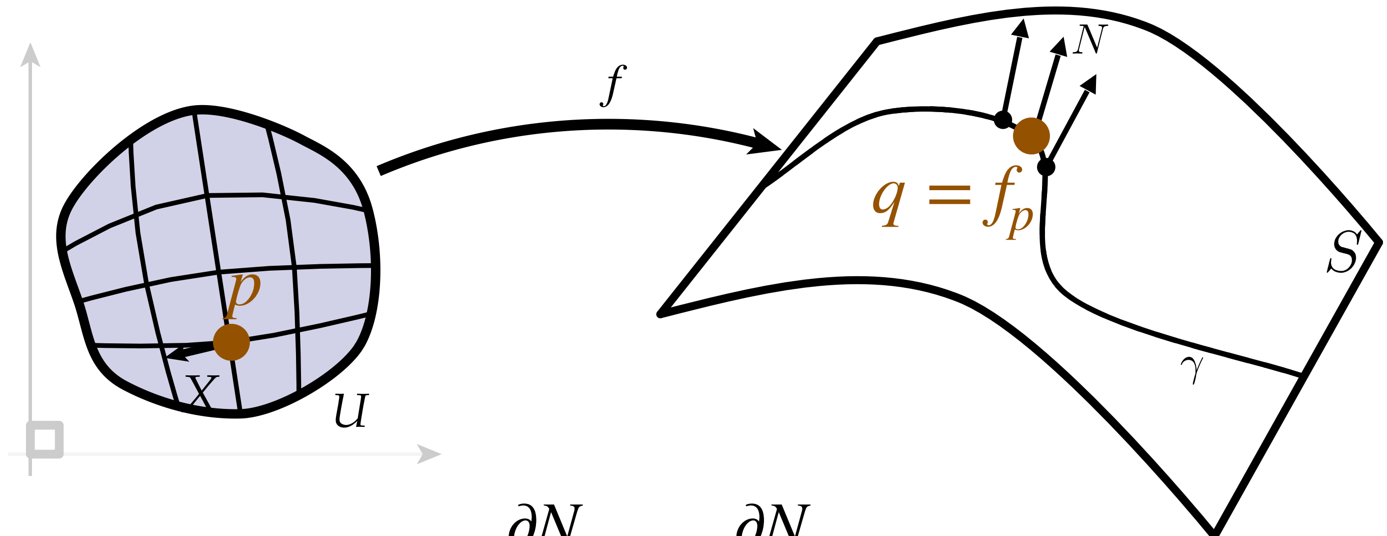
$$\dot{N}(s) \perp N(s)$$



Local change of normal is always in the tangent plane!

http://mathworld.wolfram.com/images/eps-gif/UnitSphere_800.gif

Differential of Normal



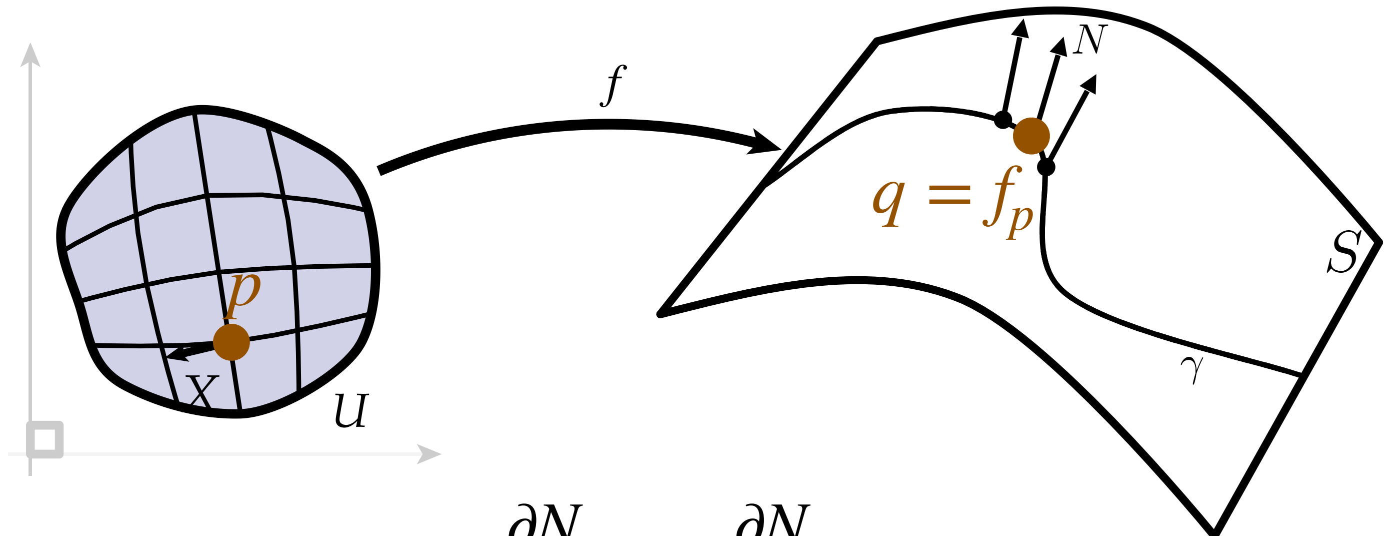
Total differential: $dN = \frac{\partial N}{\partial u} du + \frac{\partial N}{\partial v} dv$

If point $p \in \mathbb{R}^2$ moves with velocity $X = [u, v]^T$ by ϵ, the movement of N_p is:

$$\Delta N_p = \frac{\partial N}{\partial u}(\epsilon u) + \frac{\partial N}{\partial v}(\epsilon v) = \epsilon \left[\frac{\partial N}{\partial u}, \frac{\partial N}{\partial v} \right] \begin{bmatrix} u \\ v \end{bmatrix} = \epsilon [DN_p] X$$

$$DN_p := \left[\frac{\partial N}{\partial u}, \frac{\partial N}{\partial v} \right] \in \mathbb{R}^{3 \times 2}$$

Differential of Normal



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$$DN_p := \left[\frac{\partial N}{\partial u}, \frac{\partial N}{\partial v} \right] \in \mathbb{R}^{3 \times 2}$$

Note: $[DN_p]X \in \mathbf{T}_p(\mathbb{R}^3)$

Curvature $\overrightarrow{\kappa}$ of γ at p

- Recall we need the arc-length parameterization and measure the change of normal
- Recall that tangent vector $\|\mathbf{T}\| = 1$ under arc-length parameterization. So we need to scale X by μ so that:

$$\|Df_p[\mu X]\| = 1 \quad \Longrightarrow \quad \mu = \frac{1}{\|Df_p X\|}$$

- As p moves with velocity μX, the tangent is

$$Df_p[\mu X] = \frac{Df_p X}{\|Df_p X\|}$$

- the velocity of normal change is:

$$DN_p[\mu X] = \frac{DN_p X}{\|Df_p X\|}$$

Curvature $\overrightarrow{\kappa}$ of γ at p

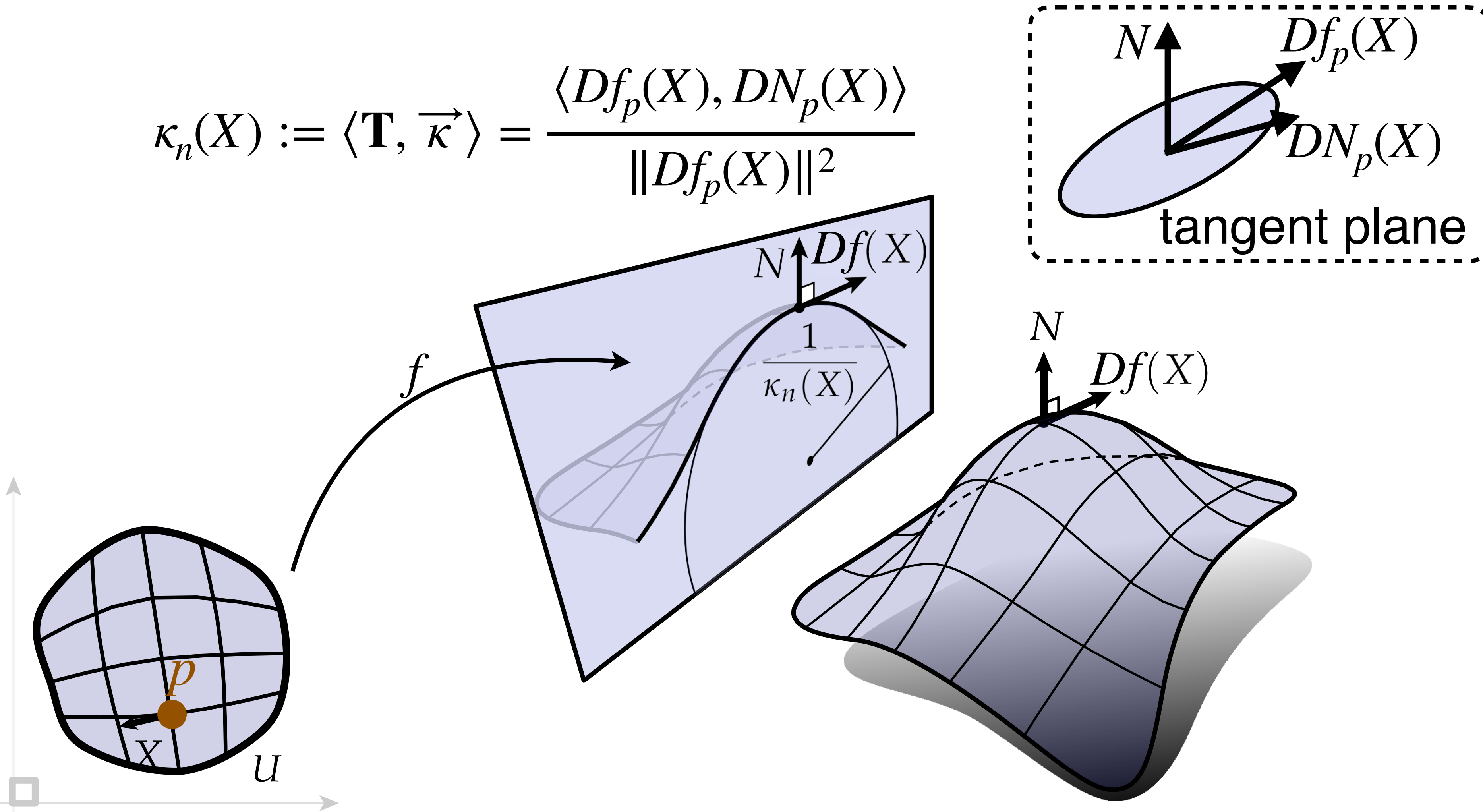
- The velocity of normal change is:

$$DN_p[\mu X] = \frac{DN_p X}{\|Df_p X\|}$$

- We denote this quantity as $\overrightarrow{\kappa}$ in this lecture (note that κ in the last lecture is a scalar, the norm of this vector)

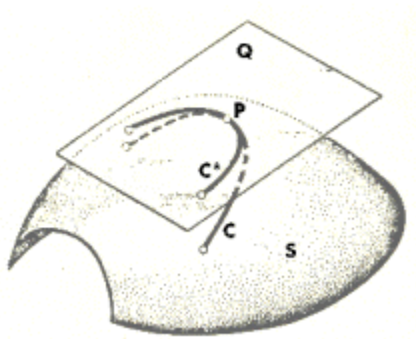
Directional Normal Curvature

$$\kappa_n(X) := \langle \mathbf{T}, \vec{\kappa} \rangle = \frac{\langle Df_p(X), DN_p(X) \rangle}{\|Df_p(X)\|^2}$$



Note: κ_n is not the curvature κ of γ

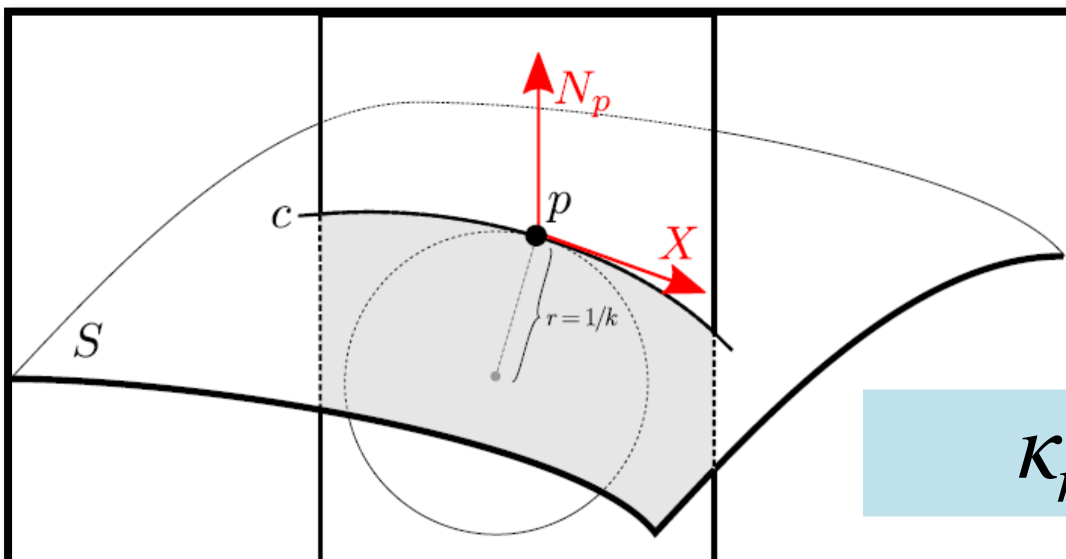
Relationship to Curvature of Curves



http://www.solitaryroad.com/c335.html

$$\kappa_g := \langle \overrightarrow{\kappa}, \mathbf{N} \times \mathbf{T} \rangle$$

(Geodesic curvature)



Drawing by Adrian Butscher

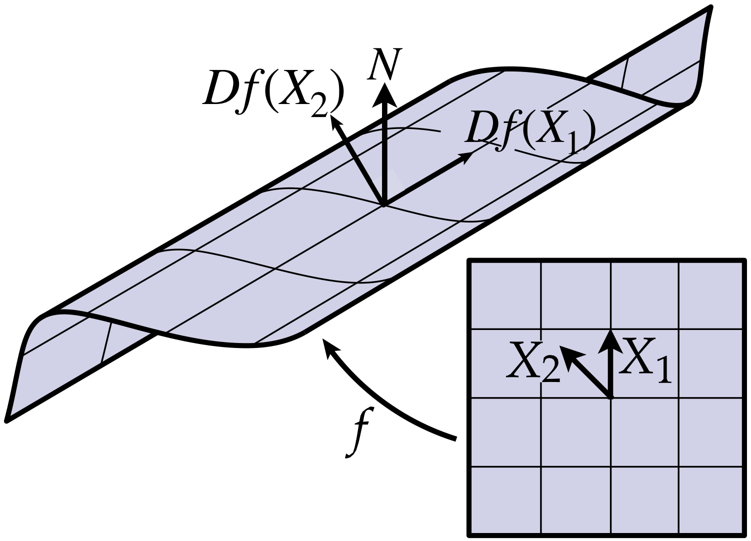
$$\kappa_n := \langle \overrightarrow{\kappa}, \mathbf{T} \rangle$$

Example

Consider a nonstandard parameterization of the cylinder (sheared along z):

$$f(u,v) := [\cos(u), \sin(u), u+v]^T \qquad Df = \begin{bmatrix} -\sin(u) & 0 \\ \cos(u) & 0 \\ 1 & 1 \end{bmatrix}$$

$$N = \begin{bmatrix} \cos(u) \\ \sin(u) \\ 0 \end{bmatrix} \qquad DN =$$



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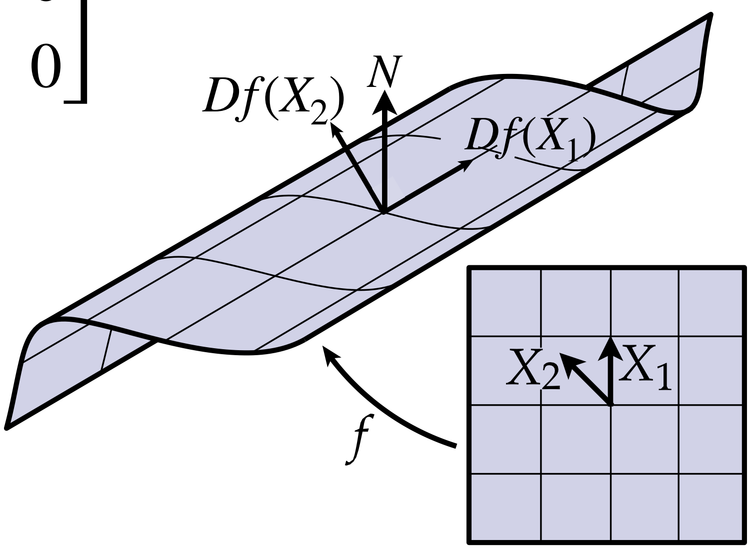

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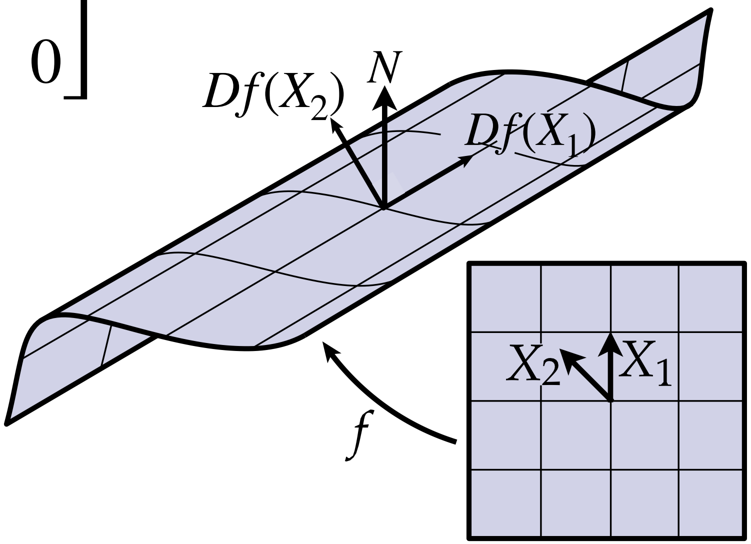
$$X_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad X_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$



$$\kappa_n(X_1) =$$

$$\kappa_n(X_2) =$$

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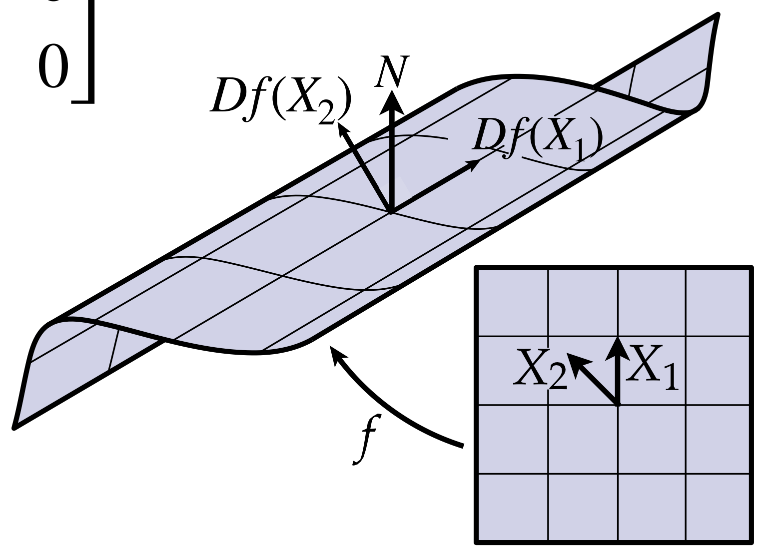
$$X_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad X_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\kappa_n(X_1) = \frac{\langle Df(X_1), DN(X_1)\rangle}{\|Df(X_1)\|^2} = 0$$

$$\kappa_n(X_2) = \frac{\langle Df(X_2), DN(X_2)\rangle}{\|Df(X_2)\|^2} = 1$$



Summary of Curvature

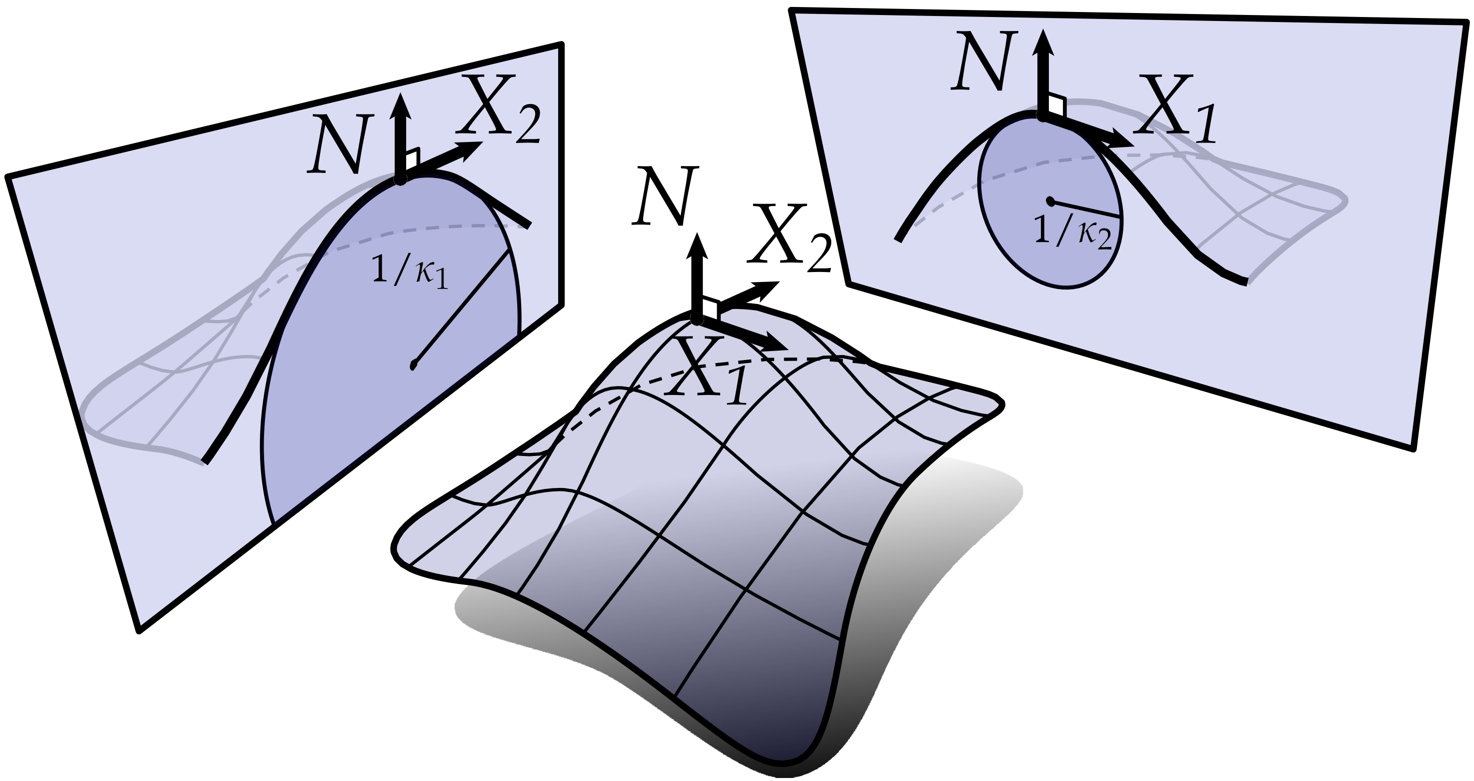
- Curvature quantifies the bending of surfaces
- Local change of normal (differential of normal) is always in the tangent plane
- Directional normal curvature quantifies how fast a surface bends along a direction

Principal Curvatures

Principal Curvatures

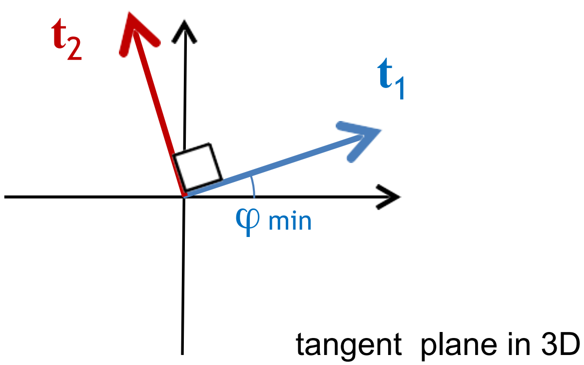
Maximal curvature: $\kappa_1 = \kappa_{\max} = \max_{\varphi} \ \kappa_n(\varphi)$

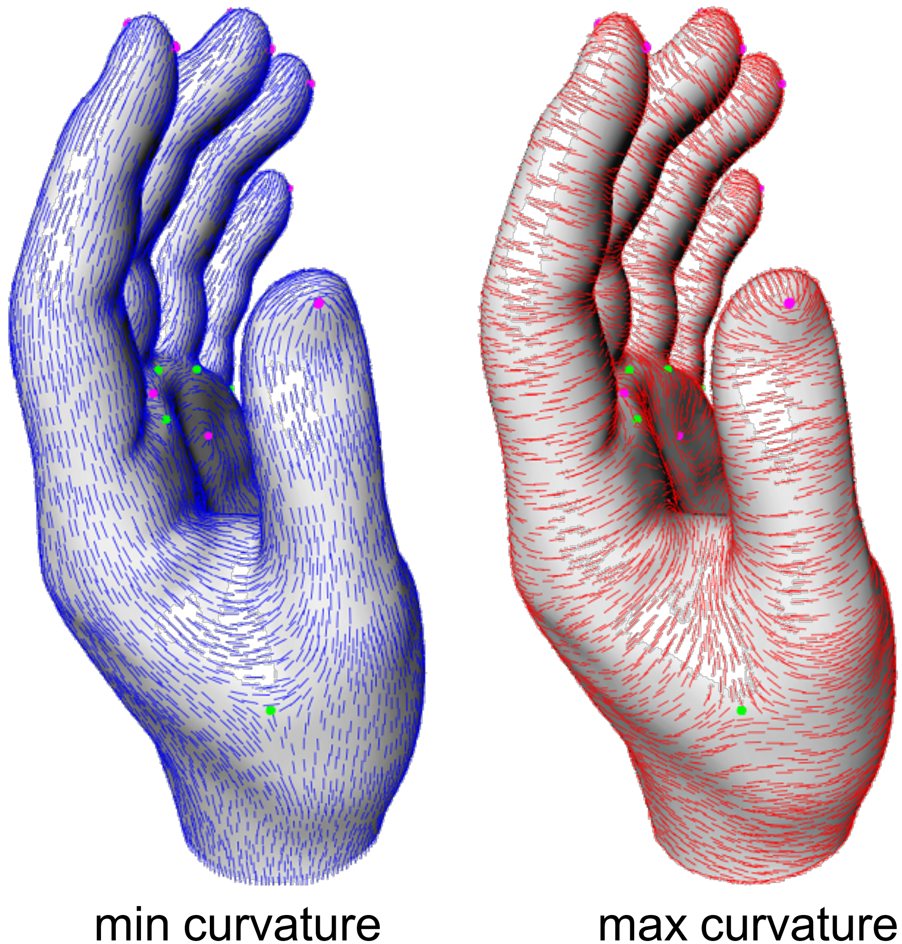
Minimal curvature: $\kappa_2 = \kappa_{\min} = \min_{\varphi} \ \kappa_n(\varphi)$



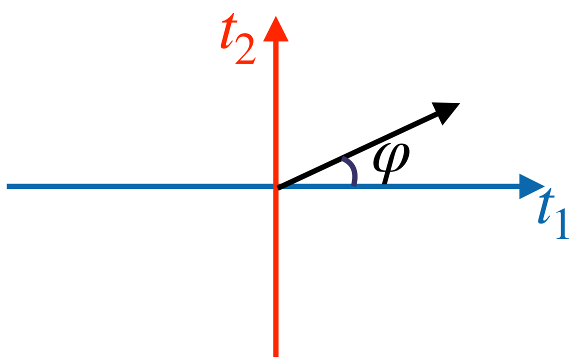
Principal Directions

Principal directions: tangent vectors corresponding to φ_{max} and φ_{min}

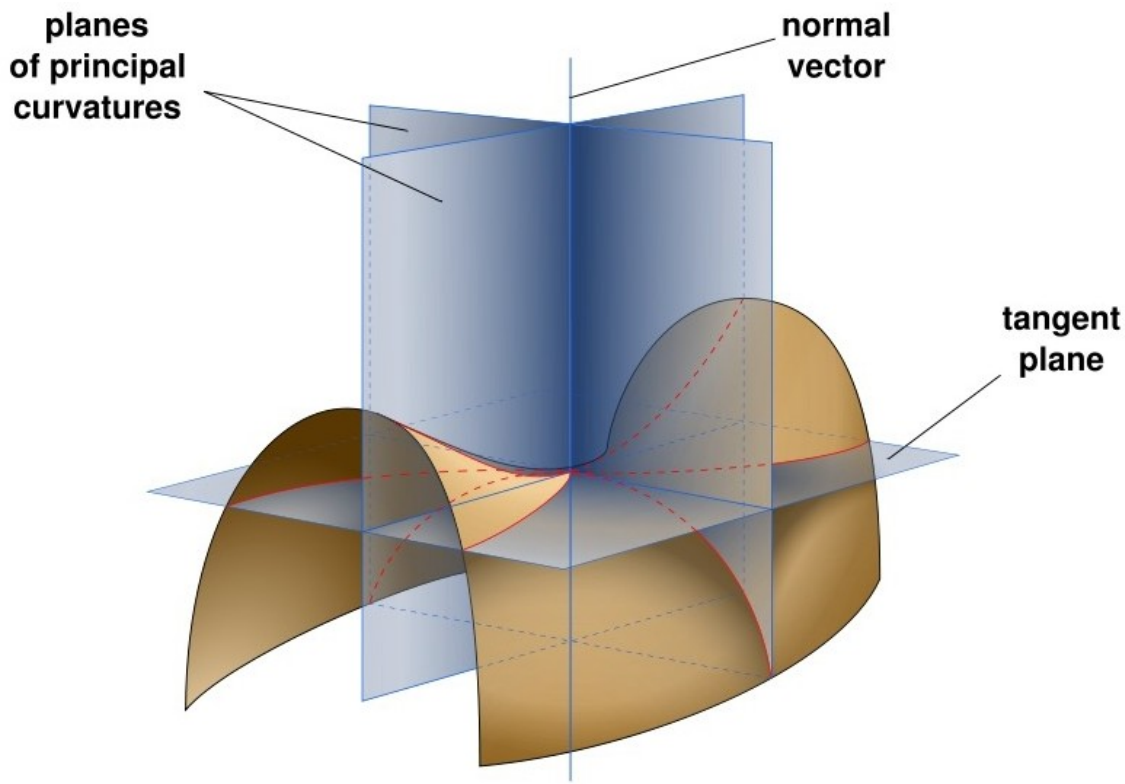




Principal Directions



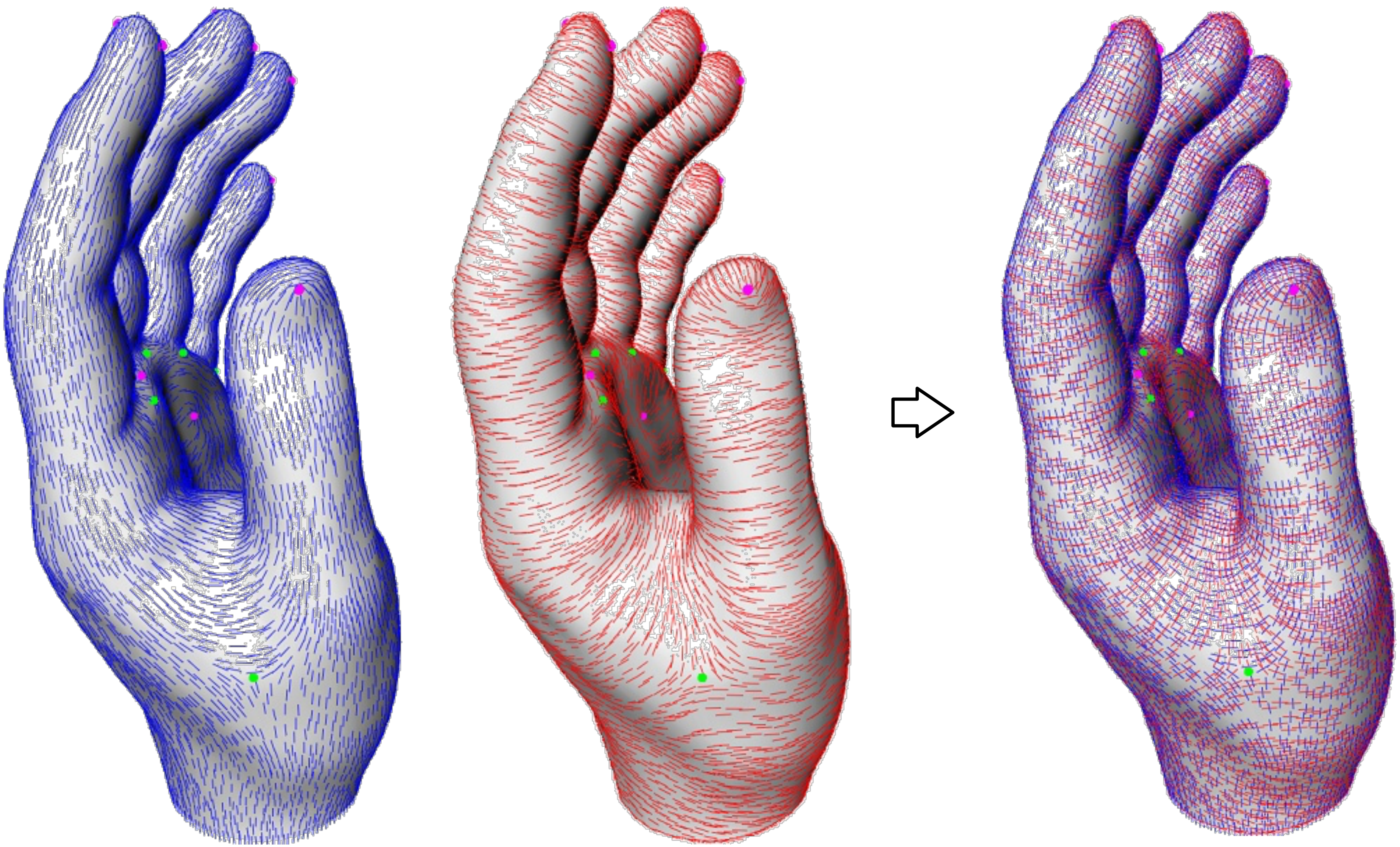
Tangent plane with principal directions as axes



Euler's Theorem: Planes of principal curvature are **orthogonal** and independent of parameterization.

$$\kappa_n(\varphi) = \kappa_1 \cos^2 \varphi + \kappa_2 \sin^2 \varphi, \qquad \varphi = \text{angle with } t_1$$

Principal Directions



Summary of Principal Curvatures

- The direction that bends fastest / slowest are principal directions, which are orthogonal to each other

- The corresponding curvatures are principal curvatures