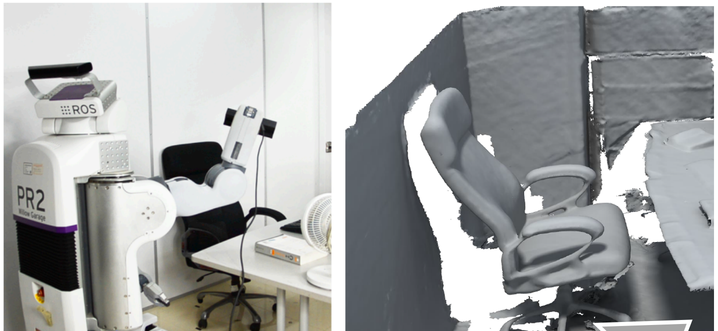
L1: Introduction

Hao Su

Agenda

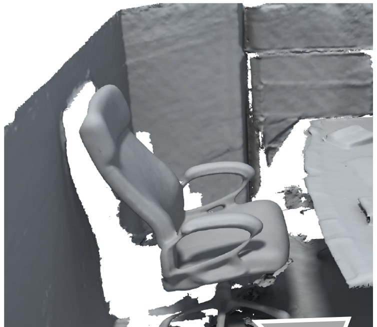
- Syllabus
- Logistics
- Curve Theory

Geometry Understanding is Important



Robotics

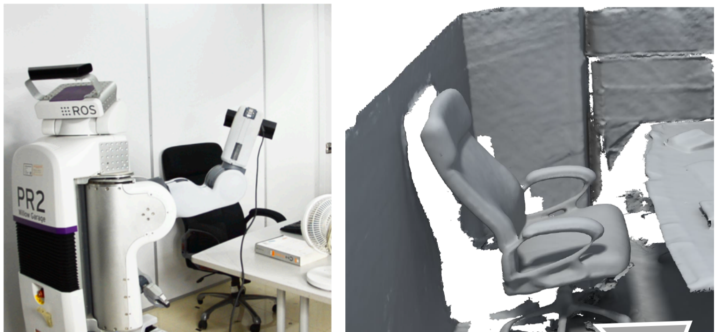
Geometry Understanding is Important



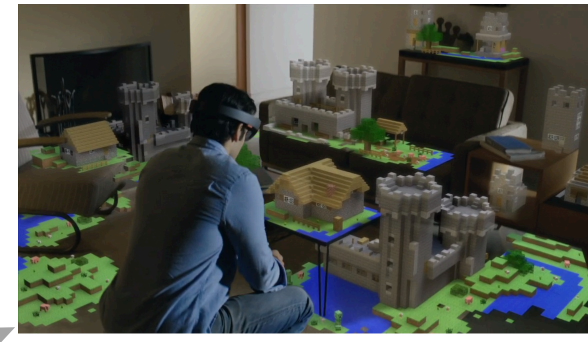
Robotics

Augmented Reality

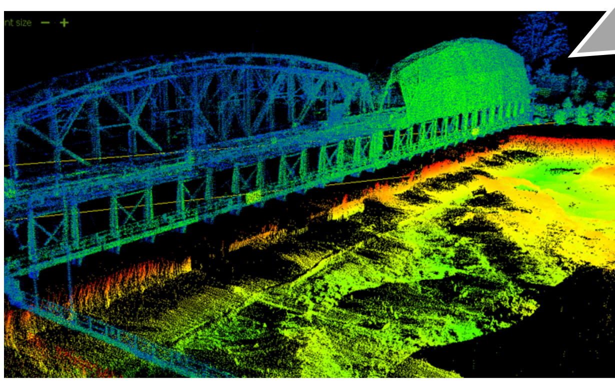
Geometry Understanding is Important



Robotics

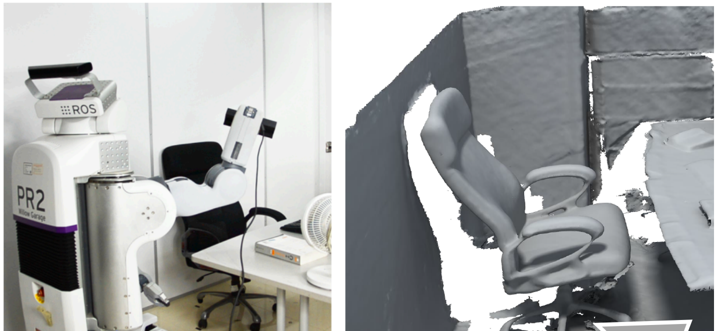


Augmented Reality



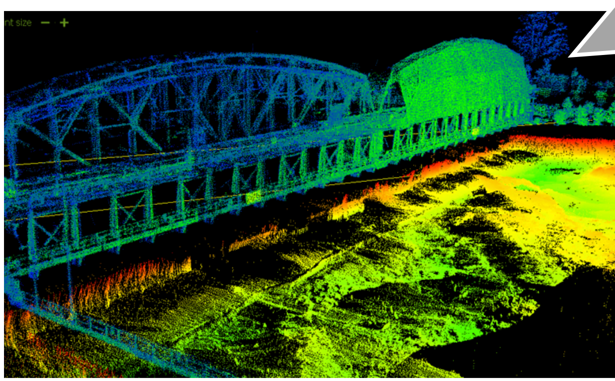
Autonomous driving

Geometry Understanding is Important

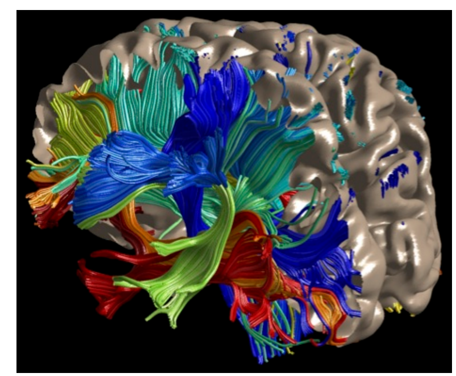


Robotics

Augmented Reality



Autonomous driving



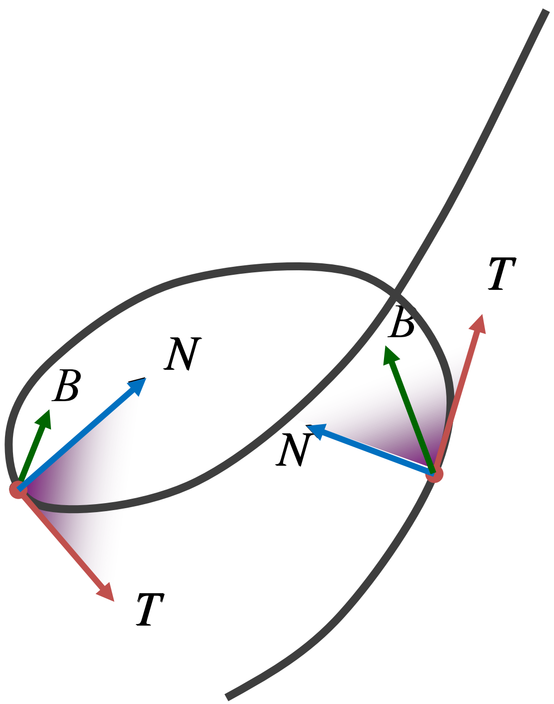
Medical Image Processing

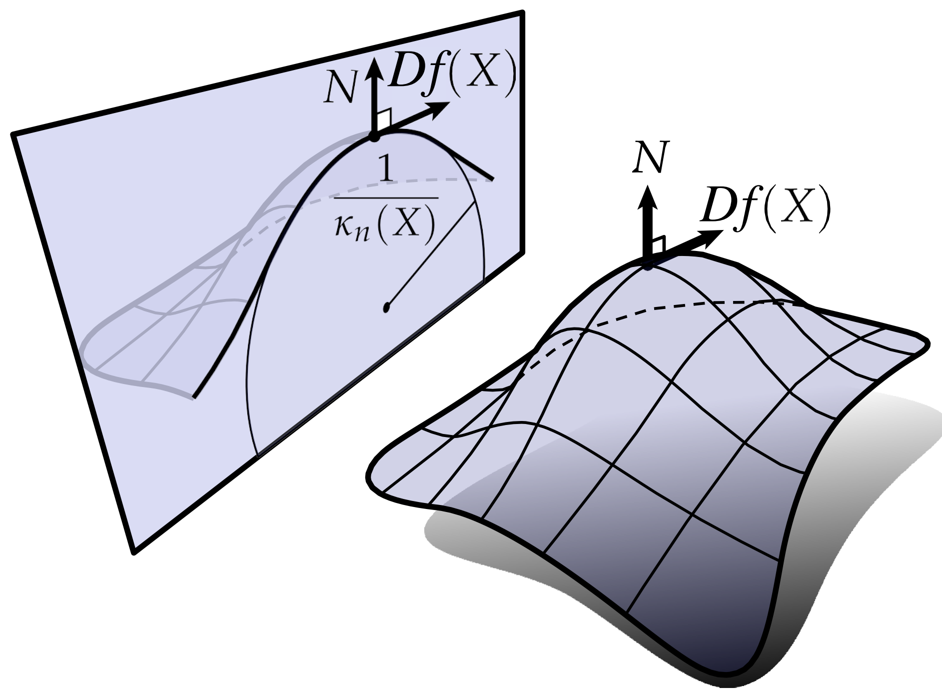
Use Machine Learning to Understand Geometries

A priori knowledge of
the 3D world

Topics Covered in This Course

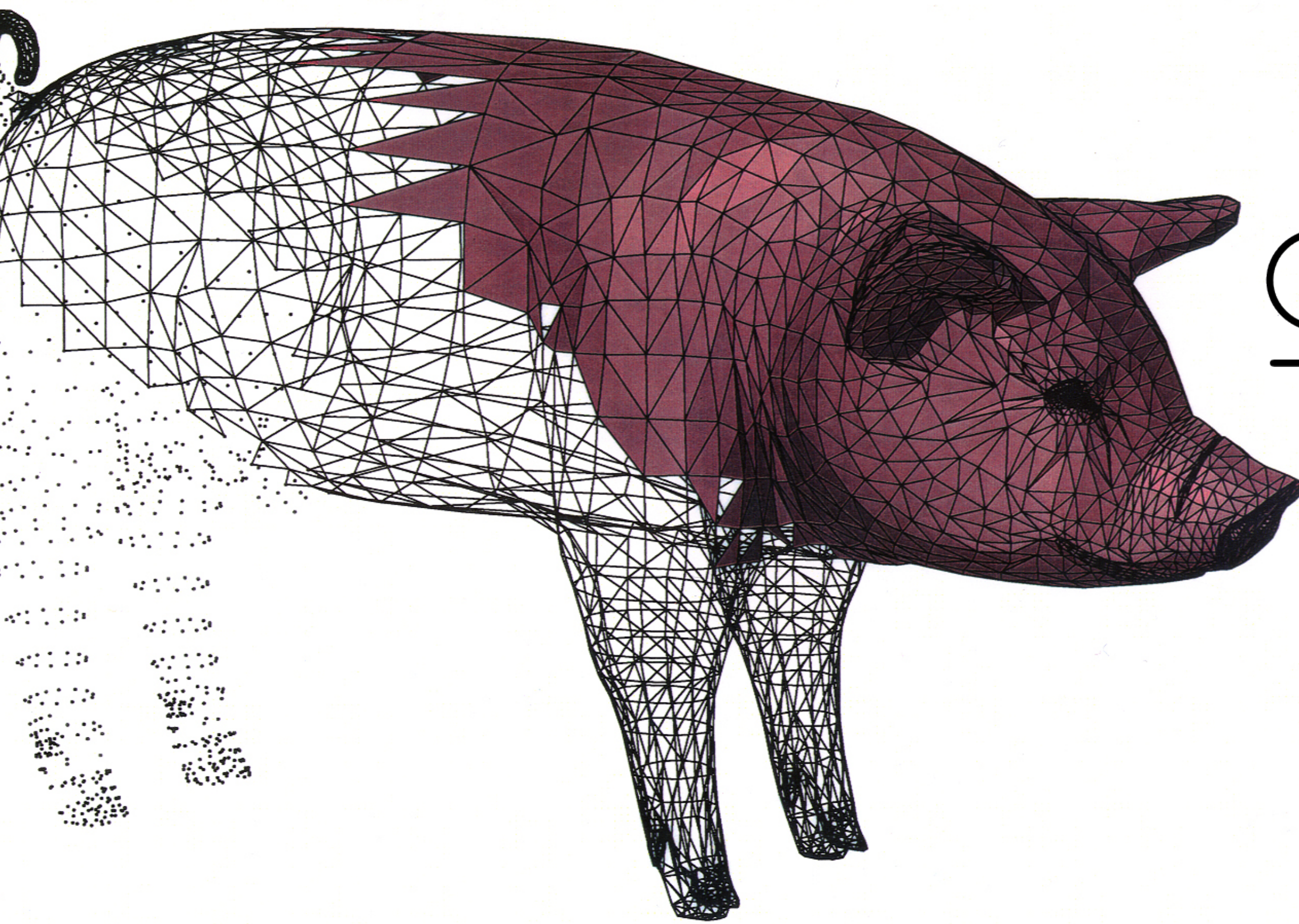
- Geometry theories





Topics Covered in This Course

- Computer Representation of Geometries



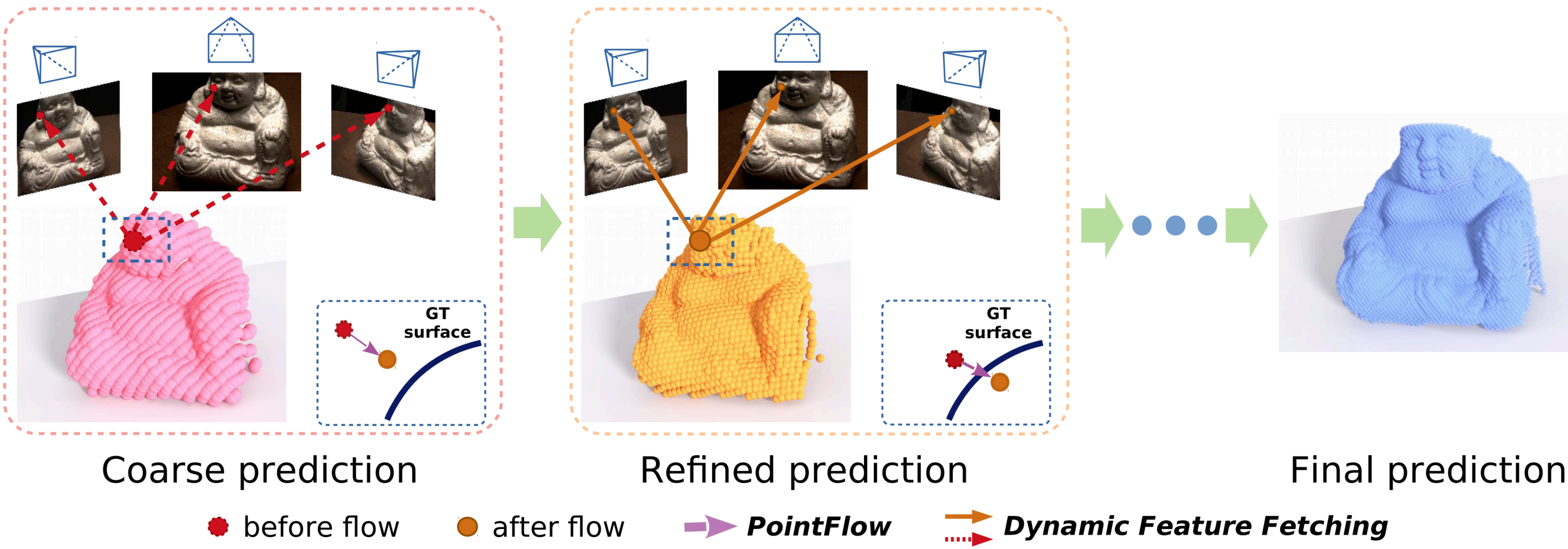
$$\subseteq \mathbb{R}^3$$

Topics Covered in This Course

- Sensing: 3D reconstruction from a single image

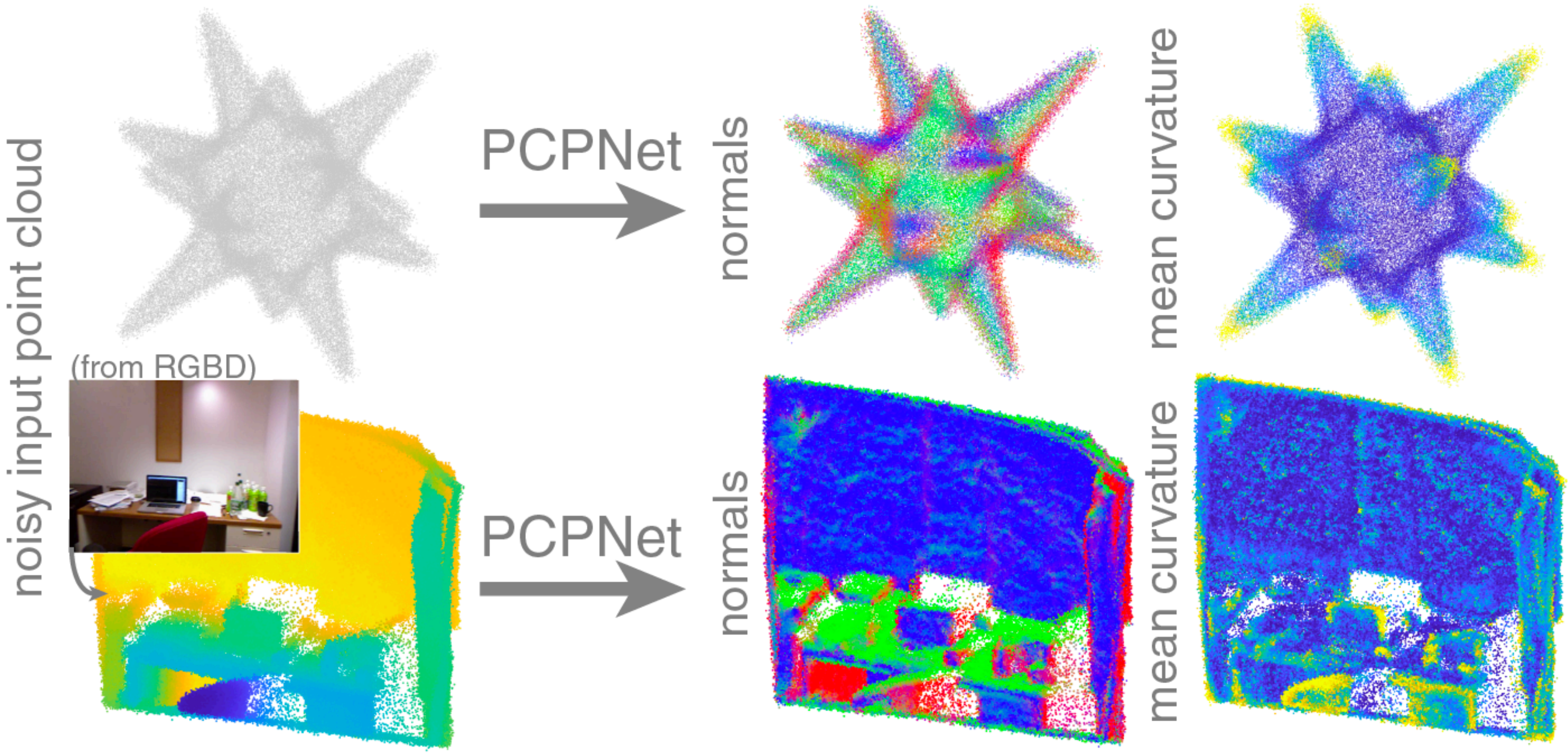
Topics Covered in This Course

- Sensing: 3D reconstruction from multiple views



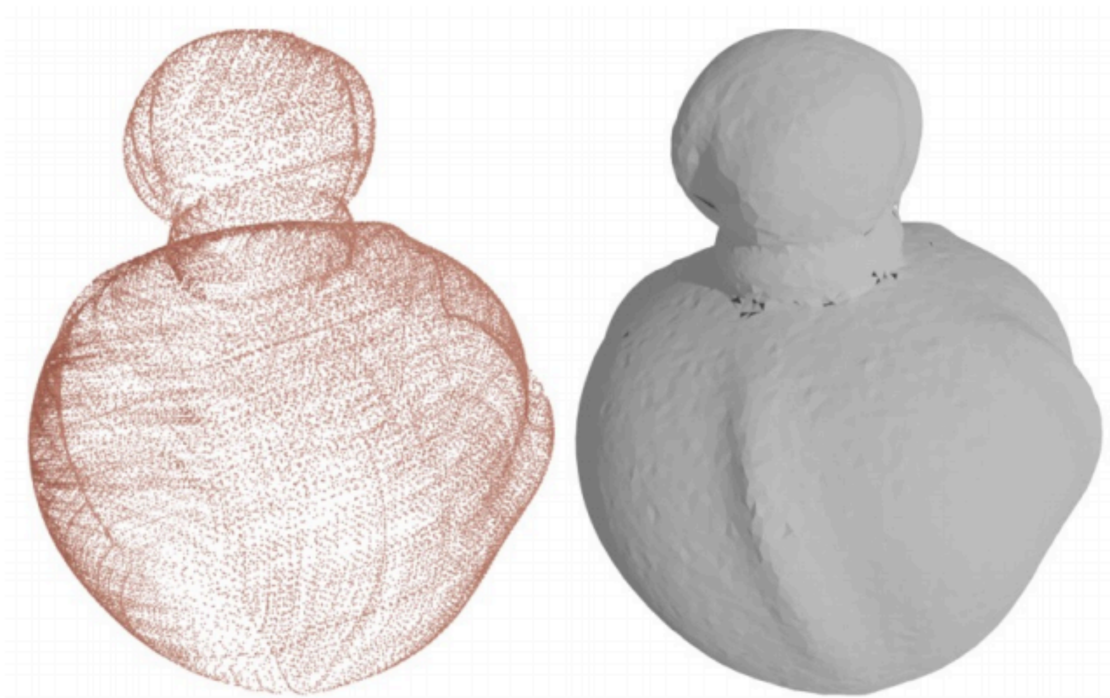
Topics Covered in This Course

- Geometry Processing: Local geometric property estimation



Topics Covered in This Course

- Geometry Processing: Surface reconstruction

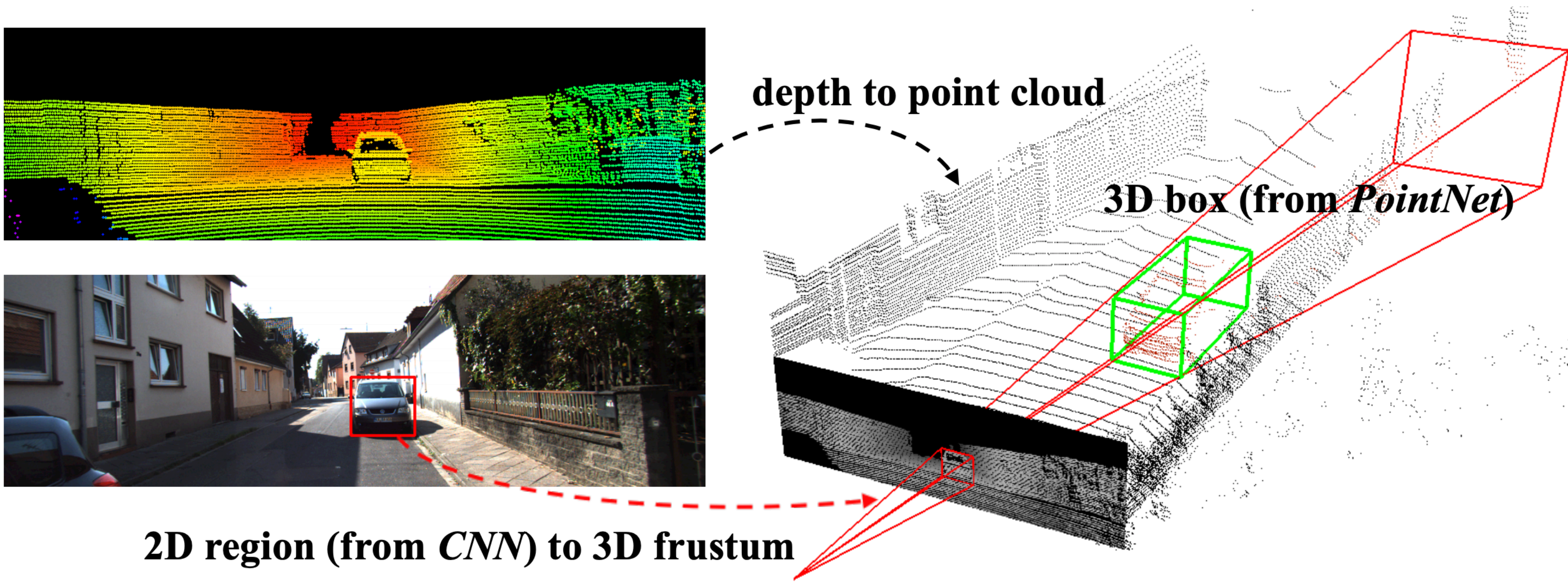


Topics Covered in This Course

- Recognition: Object classification

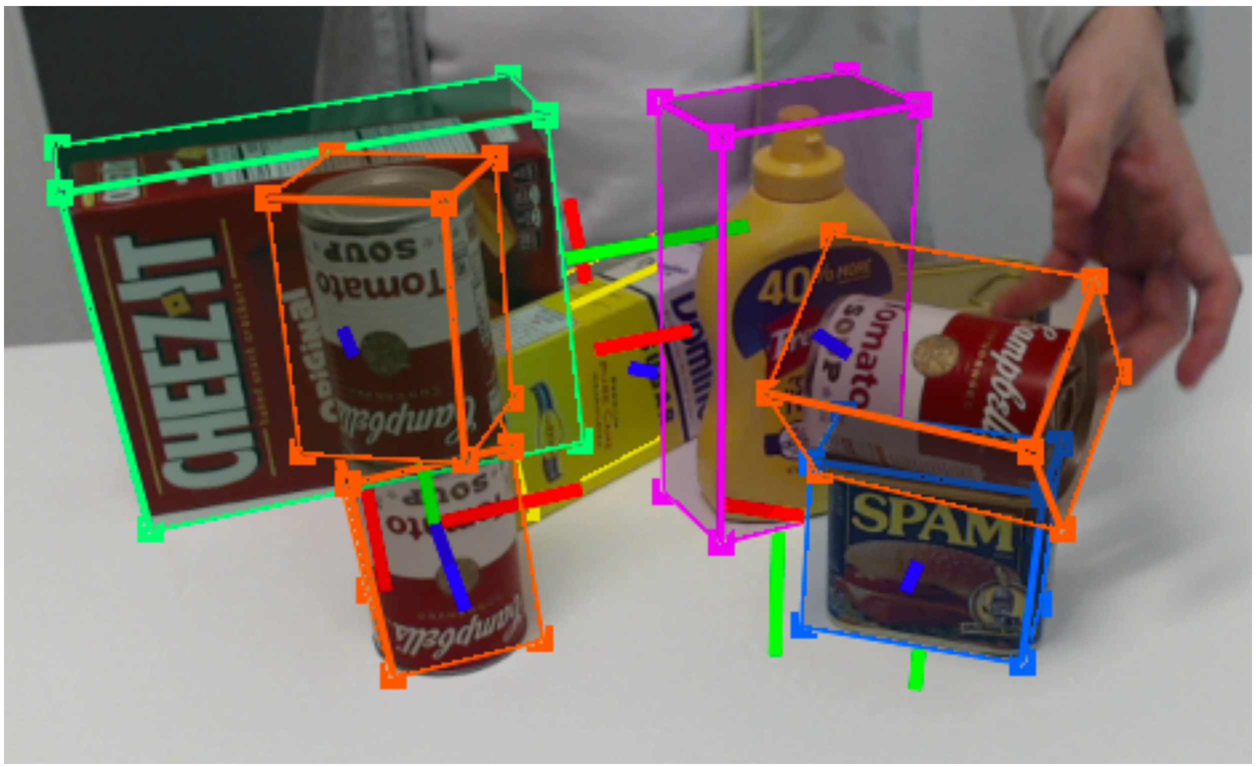
Topics Covered in This Course

- Recognition: Object detection



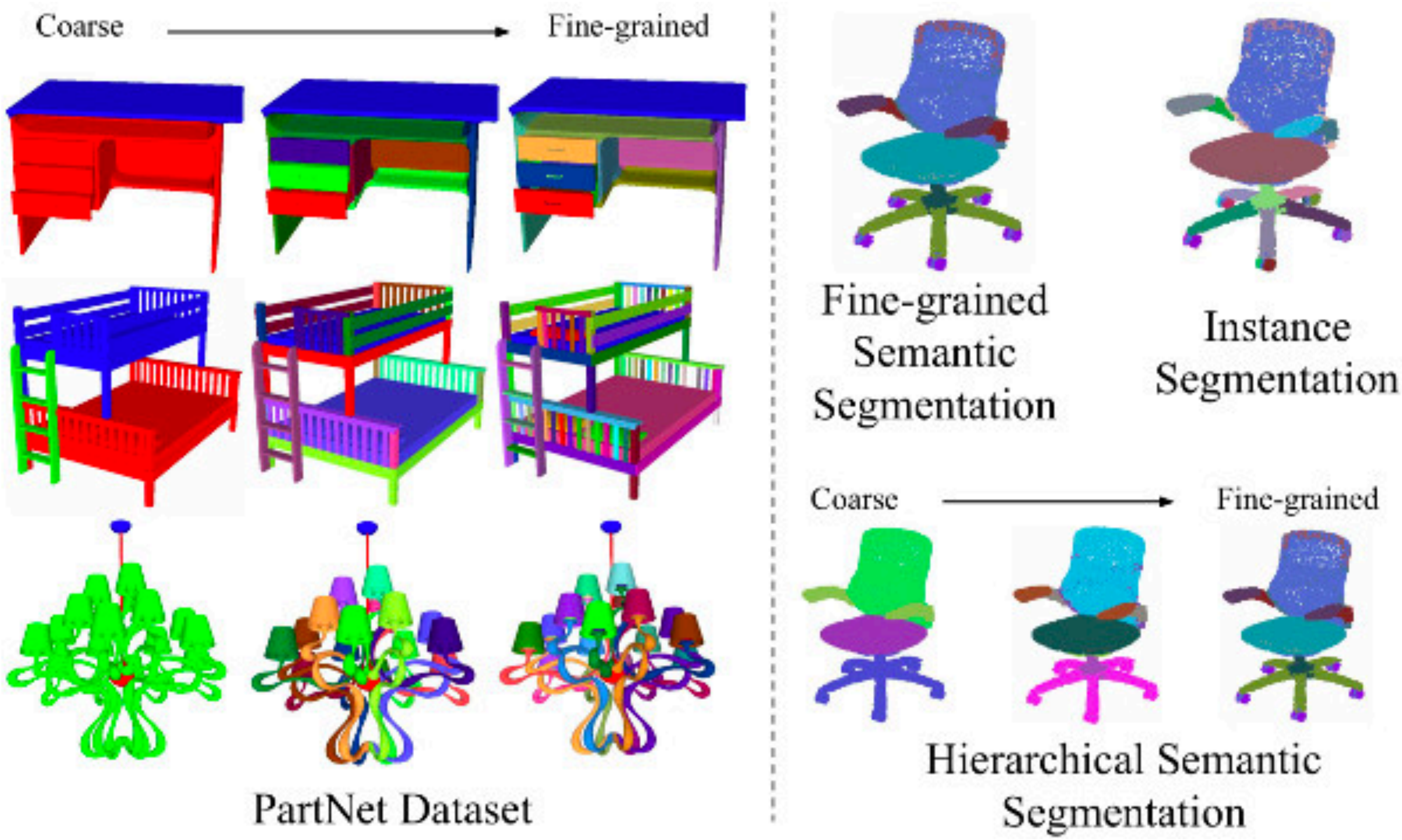
Topics Covered in This Course

- Recognition: 6D pose estimation



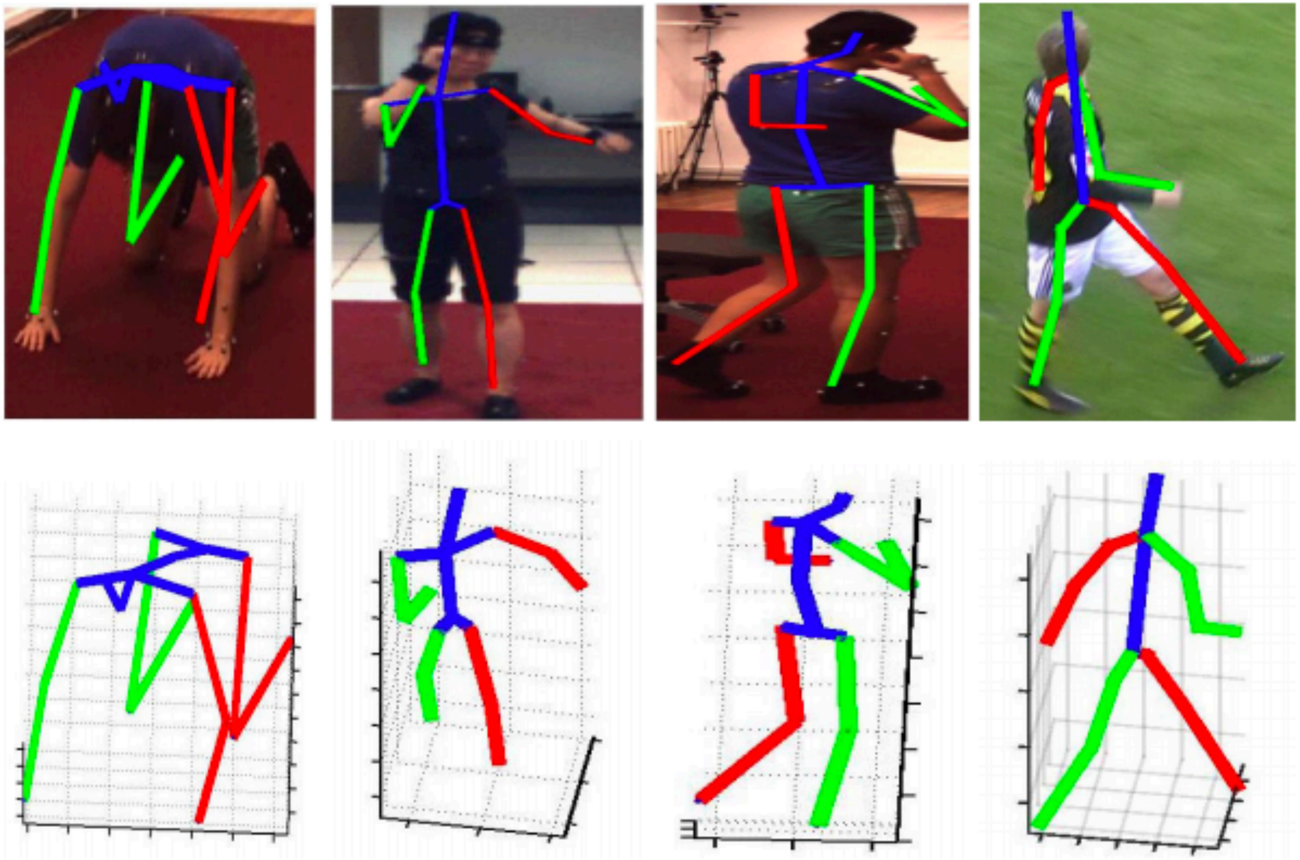
Topics Covered in This Course

- Recognition: Segmentation



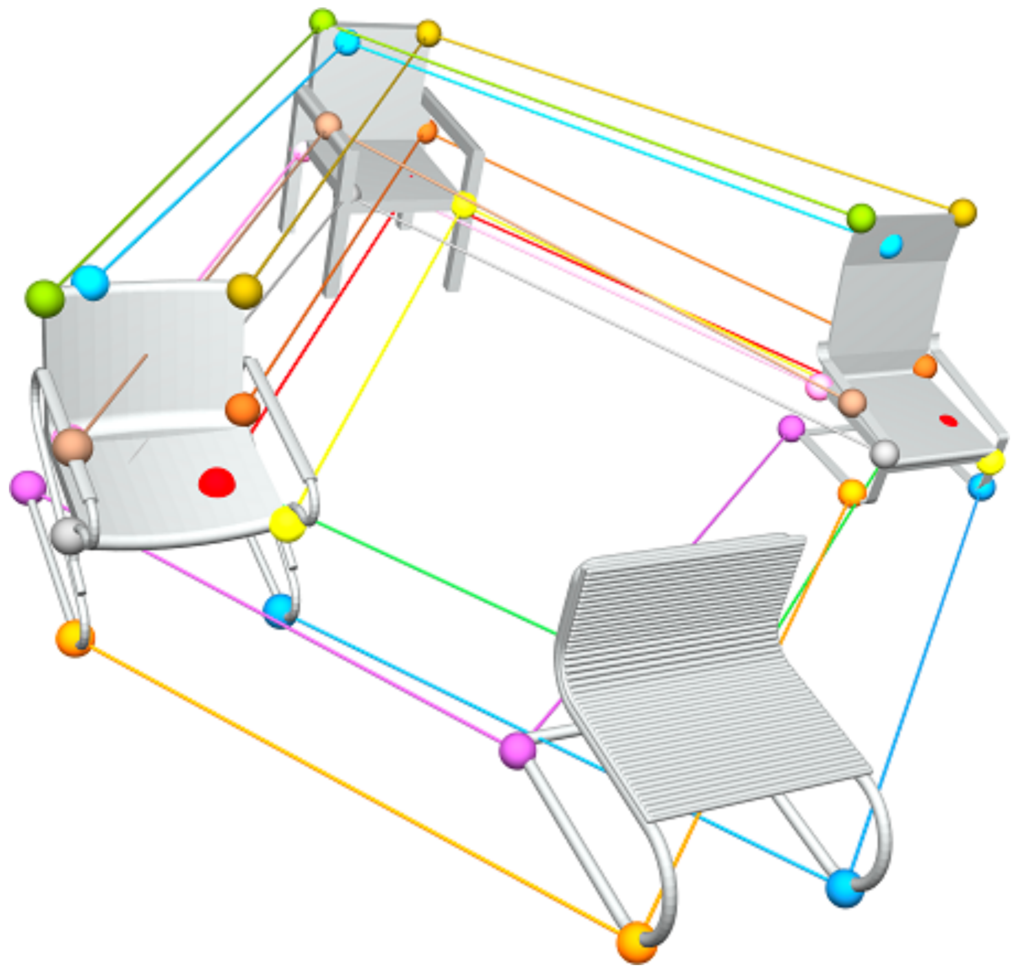
Topics Covered in This Course

- Recognition: Human pose estimation

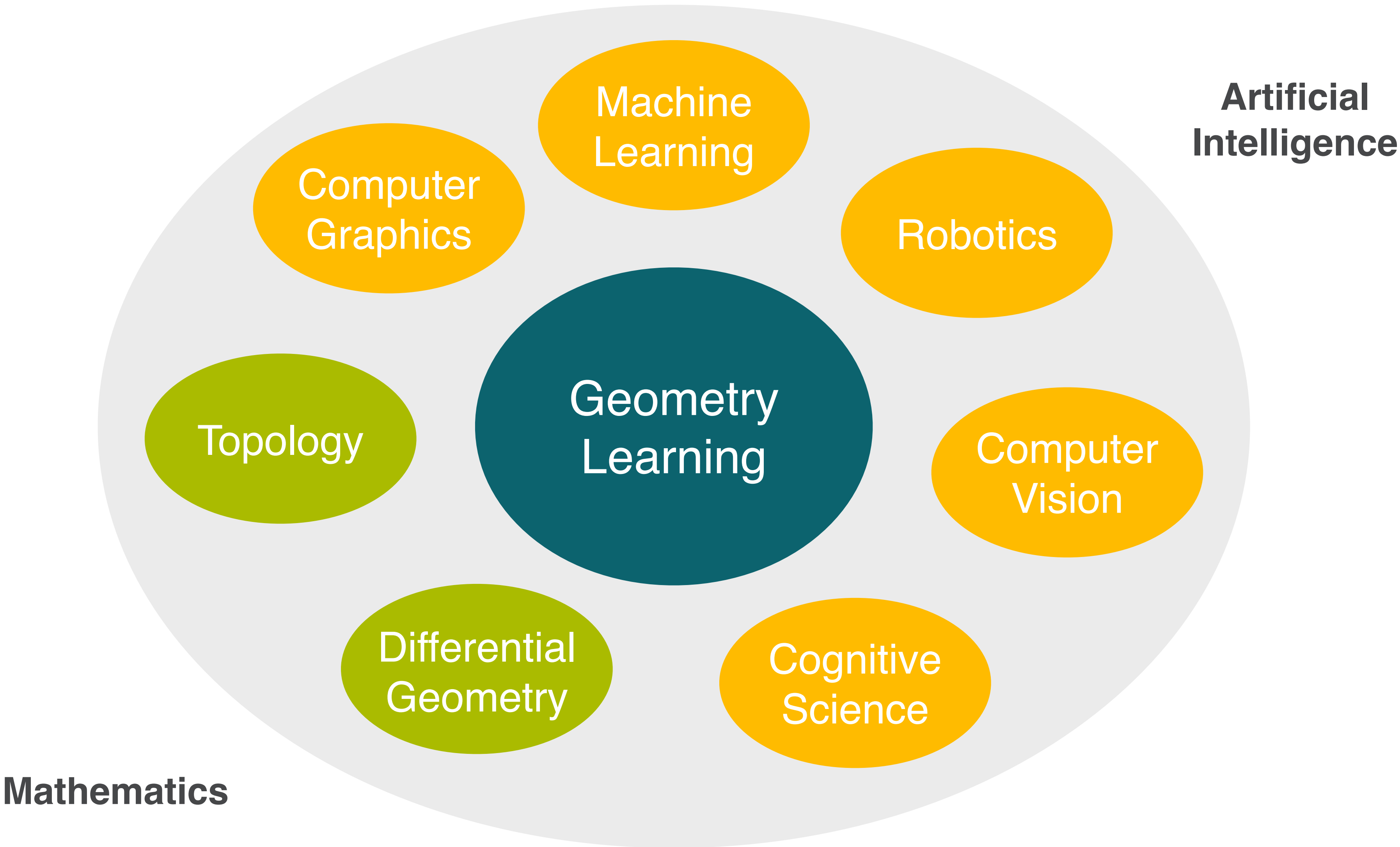


Topics Covered in This Course

- Relationship Analysis: Shape correspondences



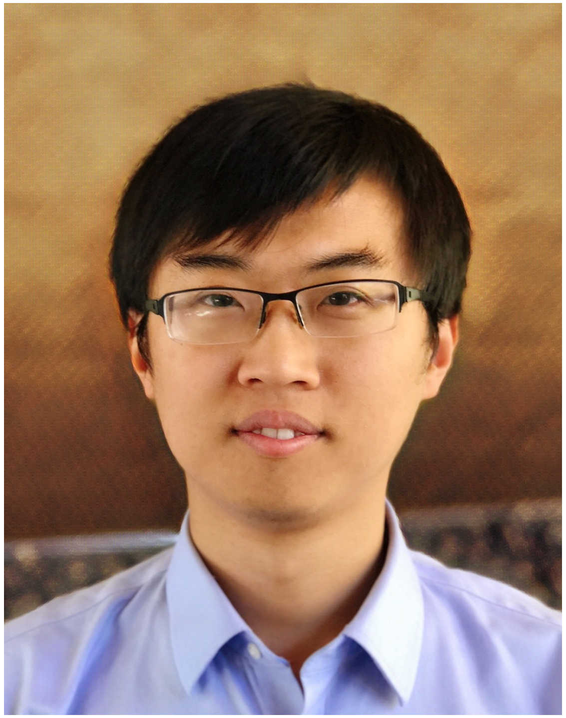
Highly Interdisciplinary Field



Course Logistic

Instructors

Instructor: Hao Su



TA: Fanbo Xiang

Teaching Goal

- State-of-the-art
  - **Enable** you to read and replicate recent 3D papers in top CV/CG conferences (not industry job oriented)
- Hands-on
  - **Heavy** programming assignments to exercise what are taught in class
- Foundational
  - Theory problems are **proof based**
  - Programming problems ask you to **implement low-level modules from scratch**

Pre-requisite: Technique

- **Skilled** in Linear Algebra
- **Familiar** with Multi-variable Calculus
- **Familiar** with Probability and Numerical Methods
- **Strong** programming skills
  - Familiar with Linux Toolchain
  - Familiar with python, numpy, and pytorch
- Course/project experiences in computer vision or deep learning

Background Check

- On Piazza now (HW0)
  - Visible to enrolled and waitlist students
- 5 points in your final grade
- **Mandatory!** We will not grade your subsequent homeworks without seeing your HW0.
- If you are in the waitlist and intend to enroll, you need to submit HW0 by this deadline
- **Due: 1/12/2021**

Pre-requisite: Resources

- This course requires deep learning resources (to run a 3D recognition challenge)

- Unfortunately, we do not have computational resources to support ~50 students

- Please find the server with the following configuration:
  >= 50G disk space
  >= 1 GPU with 10G memory

Assignments

- 4 assignments and 1 final project
  - HW0: due week 2 (5 points)
  - HW1: due week 4 (20 points)
  - HW2: due week 6 (20 points)
  - HW3: due week 8 (20 points)
  - Final project: final week (35 points)
  - No mid-term/final exams
- Extra credit for participation 5% (ask/answer questions in class, attend office hours)
- HW0-HW3: theory problems + programming
- Late policy: 15% grade reduction for each 12 hours late. No acceptance 72 hours after the due time.

3D Recognition Competition

- HW0-HW3: build individual modules
- Final project: integrate modules and test new ideas. Score by performance ranking. Online evaluation system will be set up
- We estimate **>=15 hrs per week** (out of class) solid time commitment
- We allow you to see homework (through Piazza) and attend the competition *even if you **audit** the course*

Course Resources

- Course website: https://haosulab.github.io/ml-meets-geometry/WI21/index.html (Google “Hao Su” -> Prof. Homepage -> Teaching -> this link)
  - Collaboration policy
  - Lecture slides
  - Office hour and location

- Piazza
  - Homework/Solution release
  - Discussions

Questions?

Curve

- Definition of curve
- Describing the shape of curves by calculus

Parameterized Curves

Intuition:

- A particle is moving in space
- At time t its position is given by

$$\gamma(t) = (x(t), y(t))$$

Example

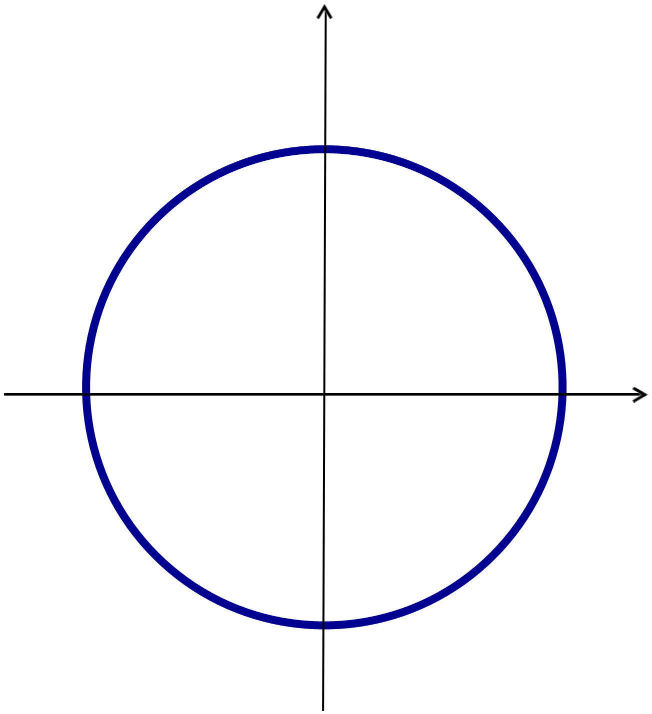
Explicit curve/circle in 2D

$$\mathbf{p} : \mathbb{R} \to \mathbb{R}^2$$

$$t \mapsto \mathbf{p}(t) = (x(t), y(t))$$

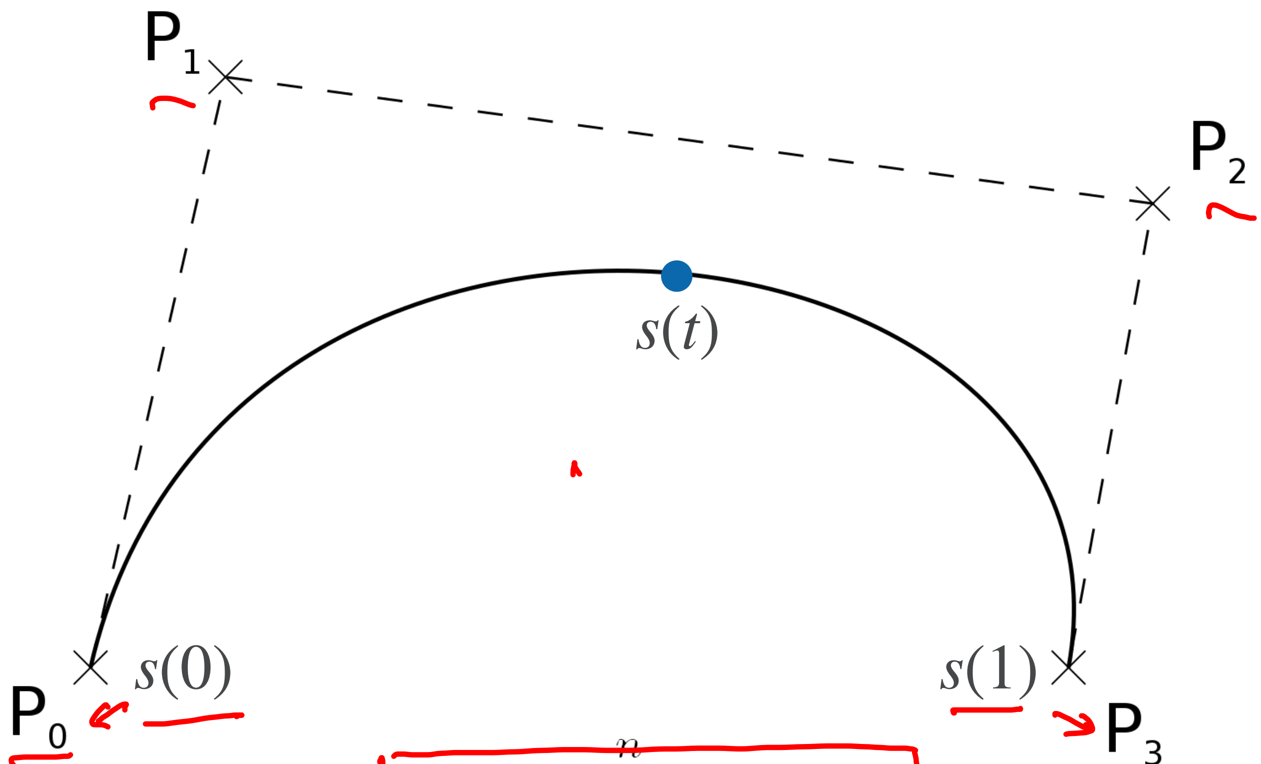
$$\mathbf{p}(t) = r\,(\cos(t), \sin(t))$$

$$t \in [0, 2\pi)$$



Application: Bezier Curves, Splines

- Smoothly “interpolate” between *a set of* points P_i
- Widely used in design (e.g., in your Powerpoint)

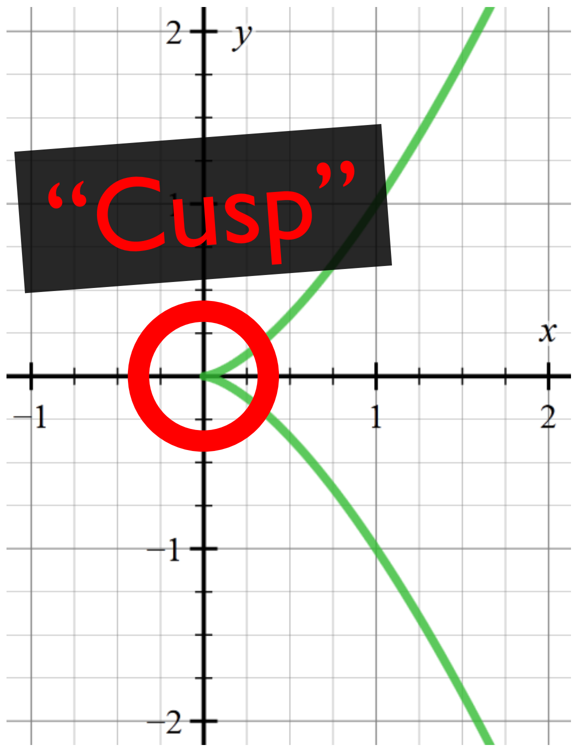


$$s(t) = \sum_{i=0}^{n} \mathbf{p}_i B_i^n(t)$$

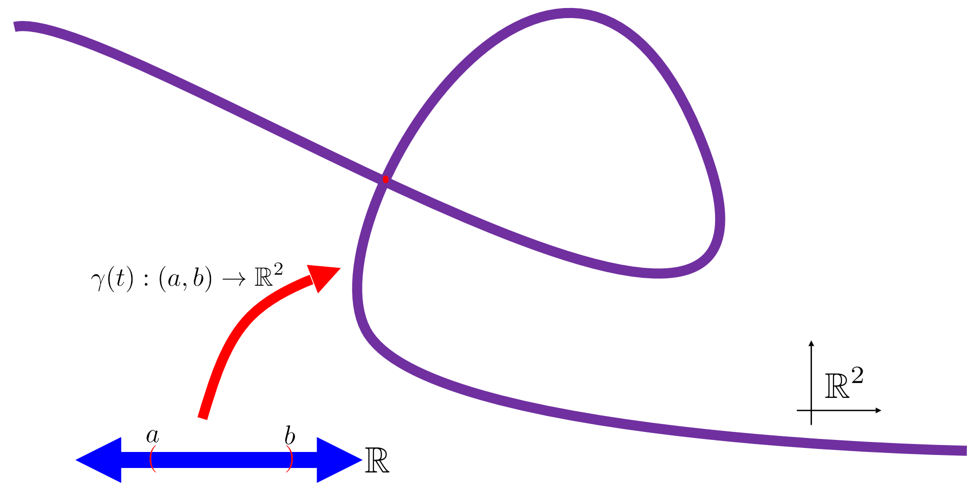
One-dimensional “Manifold”

Set of points that locally looks like a line.

Negative Examples of Manifolds



$$f(t) = (t^2, t^3)$$

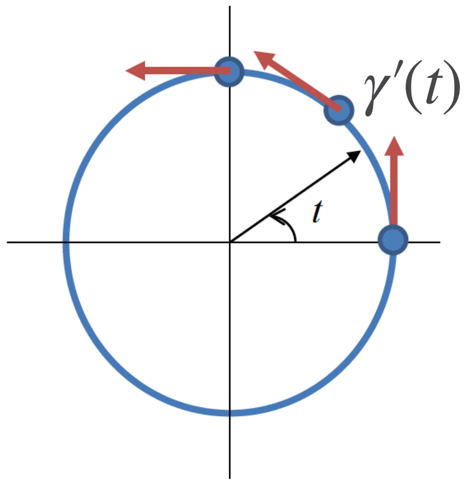


Tangent

- $\gamma'(t) = (x'(t), y'(t)) \in \mathbb{R}^2$ is the tangent vector of the curve at t

Quiz: Tangent of a Circle

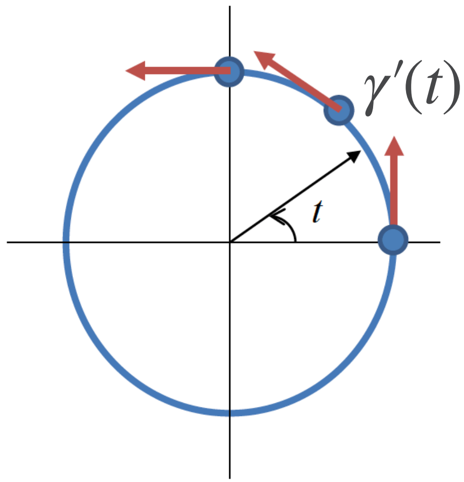
- $\gamma'(t) = (x'(t), y'(t)) \in \mathbb{R}^2$ is the tangent vector of the curve at t



$$\gamma(t) = (\cos(t), \sin(t))$$

Quiz: Tangent of a Circle

- $\gamma'(t) = (x'(t), y'(t)) \in \mathbb{R}^2$ is the tangent vector of the curve at t



$$\gamma(t) = (\cos(t), \sin(t))$$

$$\gamma'(t) = (-\sin(t), \cos(t))$$

$\gamma'(t)$ - direction of movement

$||\gamma'(t)||$ - speed of movement

Arc Length

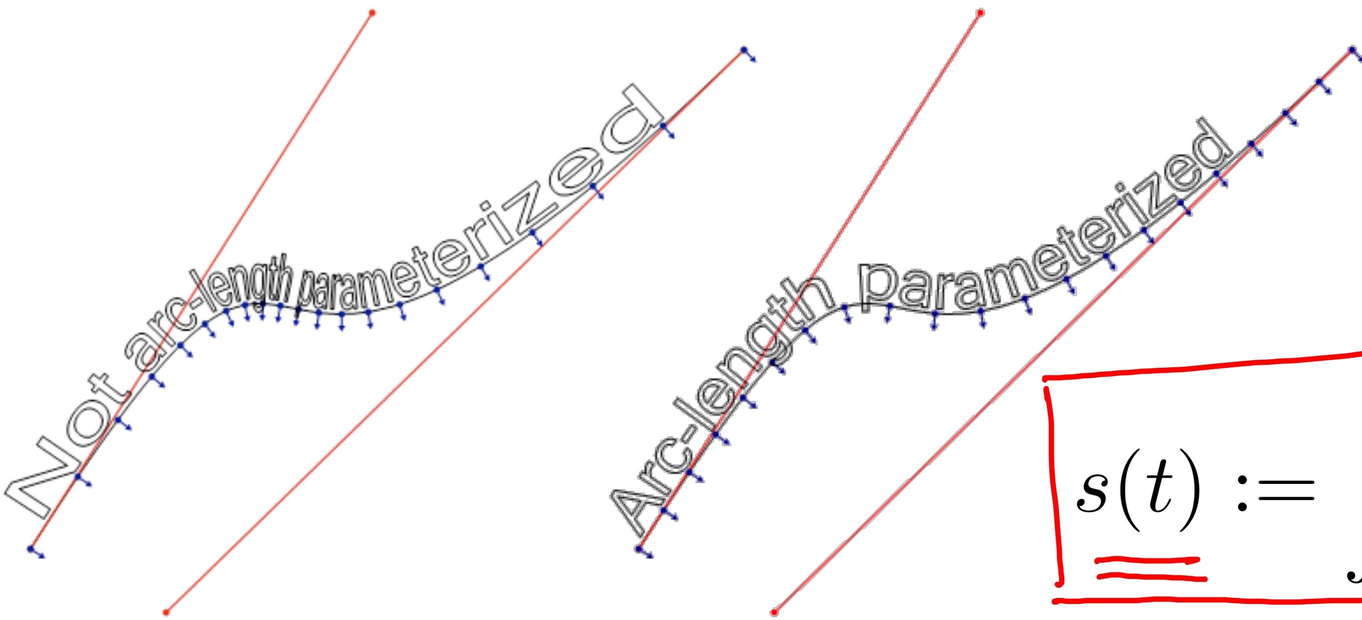
$$\int_a^b \|\gamma'(t)\|\, dt$$

speed

$\Delta t \to 0$

Parameterization by Arc Length

http://www.planetclegg.com/projects/WarpingTextToSplines.html



$$s(t) := \int_{t_0}^{t} \|\gamma'(t)\| \, dt$$

$$t(s) := \text{inverse of } s(t)$$

$$\bar{\gamma}(s) = \gamma(t(s))$$

Constant-speed parameterization

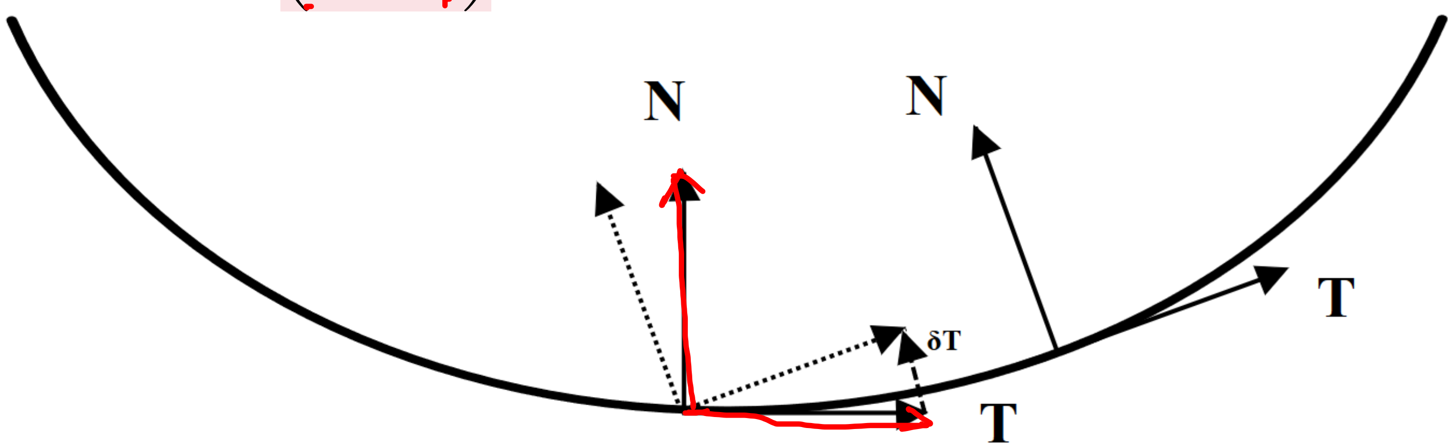
Moving Frame in 2D

$$T(s) := \gamma'(s)$$

$$\Longrightarrow \quad \text{(on board)} \; \|T(s)\| \equiv 1$$

$$N(s) := JT(s)$$

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$



https://en.wikipedia.org/wiki/Frenet%E2%80%93Serret_formulas

Lemma

$$\frac{d}{ds}\langle u(s), v(s)\rangle = \langle \frac{du}{ds}, v\rangle + \langle u, \frac{dv}{ds}\rangle$$

$(uv)' = u'v + uv'$

Derivation of $\|T(s)\| \equiv 1$

$$s(t) = \int_{t_0}^{t} \|\gamma'(t)\|\, dt$$

$$\frac{ds}{dt} = \|\gamma'(t)\|.$$

$$\|T(s)\| = \|\gamma'(s)\| = \left\|\frac{d\gamma}{ds}\right\| = \left\|\frac{d\gamma}{dt}\cdot\frac{dt}{ds}\right\| = \left\|\frac{d\gamma}{dt}\right\|\cdot\left\|\frac{dt}{ds}\right\|$$

$$t(s) = s^{-1}(t)$$

$$\frac{dt}{ds} = \frac{1}{\frac{ds}{dt}} \Rightarrow \left\|\frac{dt}{ds}\right\| = \frac{1}{\left\|\frac{ds}{dt}\right\|} = \frac{1}{\|\gamma'(t)\|}$$

$$\|T(s)\| \stackrel{\Delta}{=} \|\gamma'(t)\| \cdot \frac{1}{\|\gamma'(t)\|} = 1$$

(See notes)

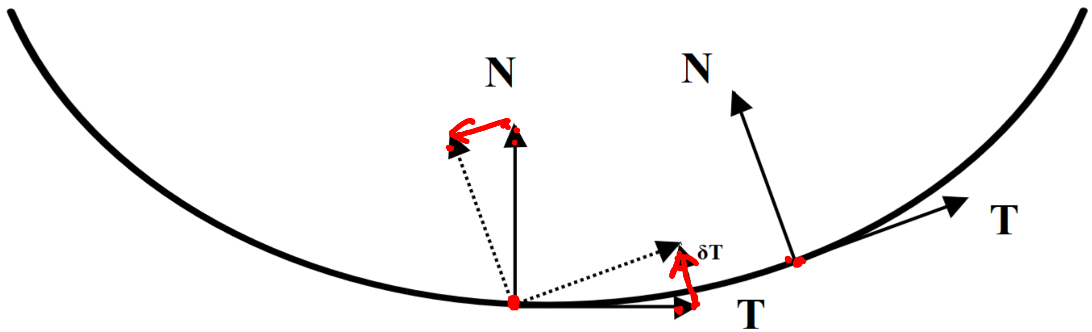
Turtles All The Way Down

$$\frac{dT}{ds} = k(s)\,N(s)$$

$$\frac{dN}{ds} = -k(s)\,T(s)$$

On the board:

$$\frac{d}{ds}\begin{pmatrix} T(s) \\ N(s) \end{pmatrix} := \begin{pmatrix} 0 & k(s) \\ -k(s) & 0 \end{pmatrix}\begin{pmatrix} T(s) \\ N(s) \end{pmatrix}$$



https://en.wikipedia.org/wiki/Frenet%E2%80%93Serret_formulas

Use coordinates *from* the curve to express its shape!

$$\frac{d}{ds}\begin{pmatrix} T(s) \\ N(s) \end{pmatrix} := \begin{pmatrix} 0 & k(s) \\ -k(s) & 0 \end{pmatrix} \begin{pmatrix} T(s) \\ N(s) \end{pmatrix}$$

① $\frac{dT}{ds} = \kappa \cdot N$. ② $\frac{dN}{ds} = -\kappa T$ ③ $\langle T, N \rangle \equiv 0$

① $\Rightarrow$ ③ to show: $\langle T, \frac{1}{\kappa}\frac{dT}{ds} \rangle \equiv 0$.

$\because \langle T, T \rangle \equiv 1$, $\Rightarrow \langle \frac{dT}{ds}, T \rangle + \langle T, \frac{dT}{ds} \rangle \equiv 0$.

$\langle \frac{dT}{ds}, T \rangle = 0$

① $\Rightarrow$ ② $\langle T, N \rangle \equiv 0$

$\langle \frac{dT}{ds}, N \rangle + \langle T, \frac{dN}{ds} \rangle \equiv 0$

$\langle T, \frac{dN}{ds} \rangle = -\langle \frac{dT}{ds}, N \rangle = -\kappa$ ④

$\langle N, N \rangle \equiv 1$.

$\langle \frac{dN}{ds}, N \rangle \equiv 0 \Rightarrow N \perp \frac{dN}{ds}$ } $\Rightarrow \frac{dN}{ds} = \alpha \cdot T$ ⑤

by ③ $N \perp T$

$\alpha = -\kappa$.

(See notes)

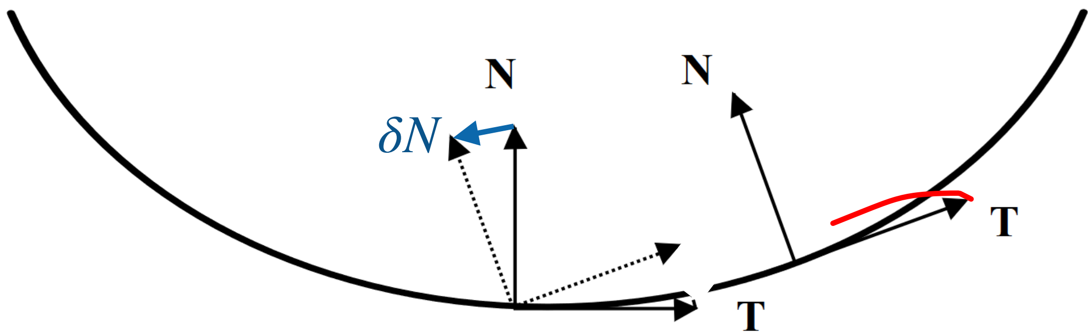
$$\frac{d}{ds}\begin{pmatrix} T(s) \\ N(s) \end{pmatrix} := \begin{pmatrix} 0 & k(s) \\ -k(s) & 0 \end{pmatrix} \begin{pmatrix} T(s) \\ N(s) \end{pmatrix}$$

(See notes)

Perspective of Normal Change

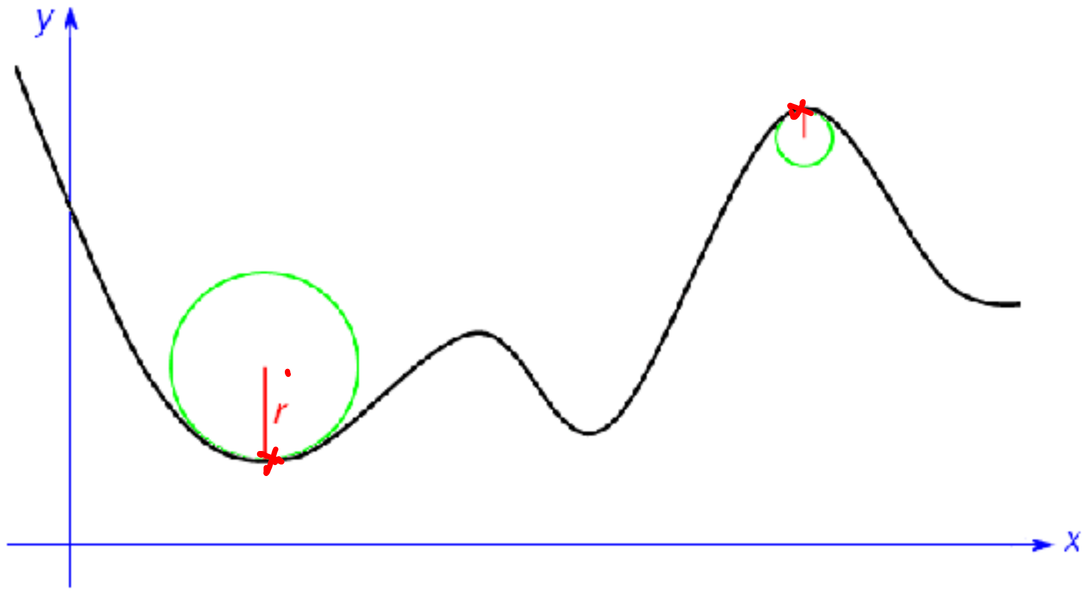
$$\mathbf{N}'(s) = -\kappa(s)\mathbf{T}(s)$$

- Curvature indicates how much the normal changes in the direction tangent to the curve



- Curvature is always positive

Radius of Curvature

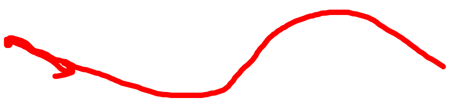


$$r(s) := \frac{1}{k(s)}$$

https://www.quora.com/What-is-the-base-difference-between-radius-of-curvature-and-radius-of-gyration

Invariance is Important

Fundamental theorem of the local theory of plane curves:



$\kappa(s)$ characterizes a **planar curve** up to rigid motion.

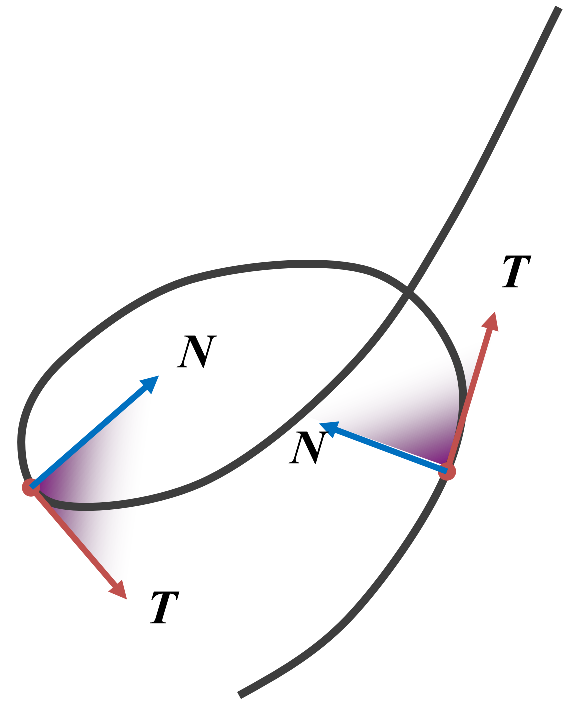
3D Curves

- Osculating Plane

$\|T\| = 1$

$N = \dfrac{\frac{dT}{ds}}{\left\|\frac{dT}{ds}\right\|}$

The plane determined by the unit tangent and normal vectors $T(s)$ and $N(s)$ is called the *osculating plane* at s



Self-reading

The Binormal Vector

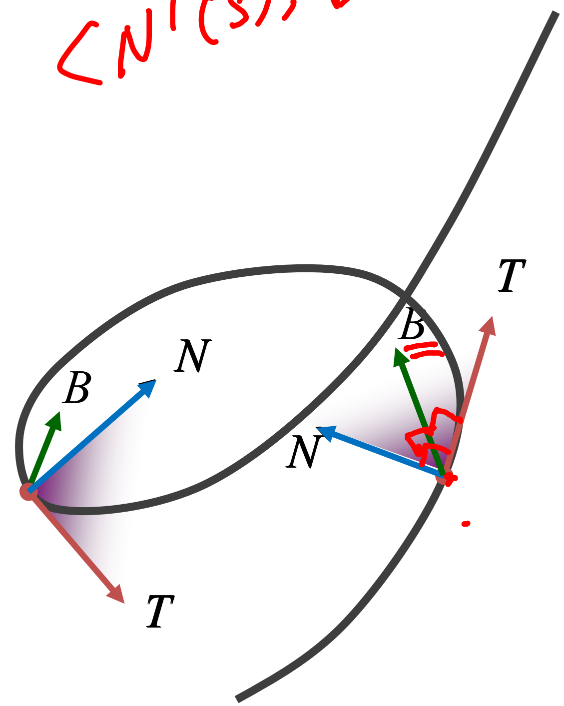
$\langle N'(s), T(s) \rangle = -\kappa$

$\langle N'(s), B(s) \rangle = \tau$

For points s, s.t. $\kappa(s) \neq 0$, the *binormal vector* $\boldsymbol{B}(s)$ is defined as:

$$\boldsymbol{B}(s) = \boldsymbol{T}(s) \times \boldsymbol{N}(s)$$

The binormal vector defines the osculating plane



Self-reading

$\mathbf{T}'(s)$

- Already used it to define the curvature:

$$\mathbf{T}'(s) = \kappa(s)\mathbf{N}(s)$$

Unit vector

- Orthogonal to $\mathbf{T}(s)$ (the same derivation as 2D curve)

- Since along the direction of $\mathbf{N}(s)$, also orthogonal to $\mathbf{B}(s)$

Self-reading

$\mathbf{N}'(s)$

We know: $\langle \mathbf{N}(s), \mathbf{N}(s) \rangle$=1

From the lemma $\longrightarrow \langle \mathbf{N}'(s), \mathbf{N}(s) \rangle = 0$

(Derivative orthogonal to itself)

We know: $\langle \mathbf{N}(s), \mathbf{T}(s) \rangle = 0$

From the lemma $\longrightarrow \langle \mathbf{N}'(s), \mathbf{T}(s) \rangle = \langle -\mathbf{N}(s), \mathbf{T}'(s) \rangle$

From the definition $\longrightarrow \kappa(s) = \langle \mathbf{N}(s), \mathbf{T}'(s) \rangle$

$\longrightarrow \langle \mathbf{N}'(s), \mathbf{T}(s) \rangle = -\kappa(s)$

Self-reading

The Torsion

- From previous slide:

  $\langle \mathbf{N}'(s), \mathbf{N}(s) \rangle = 0$

  $\langle \mathbf{N}'(s), \mathbf{T}(s) \rangle = -\kappa(s)$

The remaining component of $\mathbf{N}'(s)$ is along $\mathbf{B}(s)$ direction:

$$\langle \mathbf{N}'(s), \mathbf{B}(s) \rangle = \tau(s)$$

Now we can express $N'(s)$ as

$$\mathbf{N}'(s) = -\kappa(s)\mathbf{T}(s) + \tau(s)\mathbf{B}(s)$$

Self-reading

Perspective of Normal Change

$$\mathbf{N}'(s) = -\kappa(s)\mathbf{T}(s) + \tau(s)\mathbf{B}(s)$$

- Curvature indicates how much the normal changes in the direction tangent to the curve
- Torsion indicates how much normal changes in the direction orthogonal to the osculating plane of the curve
- Curvature is always positive but torsion can be negative

Self-reading

$\mathbf{B}'(s)$

We know: $\langle \mathbf{B}(s), \mathbf{B}(s) \rangle = 1$

From the lemma $\longrightarrow$ $\langle \mathbf{B}'(s), \mathbf{B}(s) \rangle = 0$

We know: $\langle \mathbf{B}(s), \mathbf{T}(s) \rangle = 0$, $\langle \mathbf{B}(s), \mathbf{N}(s) \rangle = 0$

From the lemma $\longrightarrow$

$\langle \mathbf{B}'(s), \mathbf{T}(s) \rangle = \langle -\mathbf{B}(s), \mathbf{T}'(s) \rangle = \langle -\mathbf{B}(s), \kappa(s)\mathbf{N}(s) \rangle = 0$

From the lemma $\longrightarrow$

$\langle \mathbf{B}'(s), \mathbf{N}(s) \rangle = \langle -\mathbf{B}(s), \mathbf{N}'(s) \rangle = -\tau(s)$

Now we express $\mathbf{B}'(s)$ as:

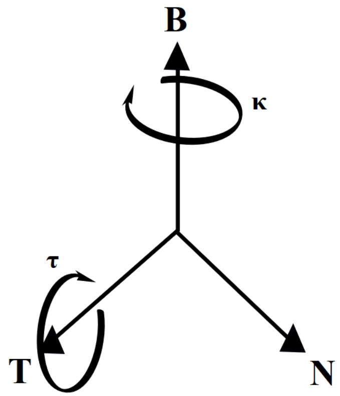
$$\mathbf{B}'(s) = -\tau(s)\mathbf{N}(s)$$

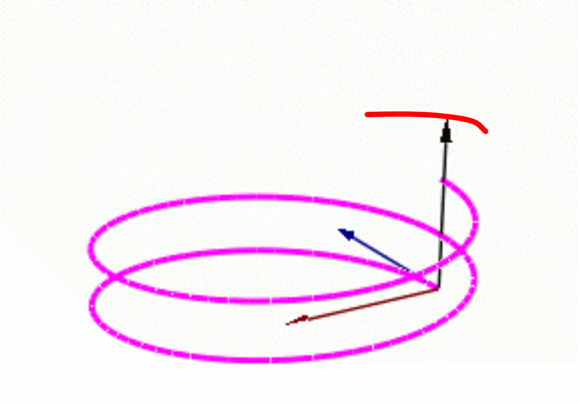
Self-reading

Frenet Frame: Curves in $\mathbb{R}^3$

- Binormal:
  - Curvature: In-plane motion
  - Torsion: Out-of-plane motion

$$\frac{d}{ds}\begin{pmatrix} T \\ N \\ B \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix}\begin{pmatrix} T \\ N \\ B \end{pmatrix}$$





Self-reading

Fundamental theorem of the local theory of space curves:

Curvature and torsion characterize a 3D curve up to rigid motion.

Summary

- Curve is a map from an interval to $\mathbb{R}^n$
- Tangent describes the moving direction
- The derivative of tangent under arc-length parameterization is normal
- Curvature (and torsion) both characterize the change of normal direction, uniquely describing the shape of a curve (up to rigid transformation)
- Tangent, normal, and binormal form a moving frame (Frenet frame)